On Decomposition of Recurrent Curvature Tensor Fields

By

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Abstract

A decomposition of curvature tensor field was studied by K. Takano [1][1] in recurrent Riemannian space. In this paper we have decomposed Berwald's curvature tensor fields, the projective tensor field and the projective curvature tensor field and derived certain theorems on it.

1. Introduction

In an n-dimensional Finsler space the curvature tensor fields due to Berwald are given by

$$(1 \cdot 1) \qquad H^i_{jk} = \partial_k \dot{\partial}_j \, G^i - \partial_j \, \dot{\partial}_k \, G^i + G^i_{kr} \, \dot{\partial}_j \, G^r - G^i_{rj} \, \dot{\partial}_k \, G^r$$

and

$$(1 \cdot 2) \qquad H^i_{jkh} = \partial_h \, G^i_{jk} - \partial_k \, G^i_{jh} + G^r_{jk} \, G^i_{rh} - G^r_{jh} \, G^i_{rk}$$

$$+ G^i_{rjh} \, \dot{\partial}_k \, G^r - G^i_{rjk} \, \dot{\partial}_h \, G^r,$$

where
$$G^i_{jkh} = \dot{\partial}_j \, G^i_{hk}.$$

The covariant derivative of $\dot{x}^i$ is identically zero.

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1. Numbers in brackets refer to the references at the end of this paper.

2. $\partial_i = \dfrac{\partial}{\partial x^i}$ and $\dot{\partial}_i = \dfrac{\partial}{\partial \dot{x}^i}.$

and

(2·6) $$\dot{\partial}_j v_l \phi_{kh} + \dot{\partial}_j v_k \phi_{hl} + \dot{\partial}_j v_h \phi_{lk} = v_l \phi_{jkh}$$
$$+ v_k \phi_{jhl} + v_h \phi_{jlk}.$$

Proof. The results (2·4) and (2·5) are obvious from (1·3) and (1·5) respectively. Differentiating (1·7) with respect to $\dot{x}^j$ and using the commutation formula (1·11), we get

(2·7) $$\dot{\partial}_j v_l H^i_{kh} = H^r_{kh} G^i_{rjl} - H^i_{rh} G^r_{kjl} - H^i_{kr} G^r_{hjl} .$$

Adding the expressions obtained by cyclic change of the indices k, h and l in (2·7), we have

(2·8) $$\dot{\partial}_j v_l H^i_{kh} + \dot{\partial}_j v_k H^i_{hl} + \dot{\partial}_j v_h H^i_{lk} = H^r_{kh} G^i_{rjl}$$

$$+ H^r_{hl} G^i_{rjk} + H^r_{lk} G^i_{rjh}.$$

By virtue of (1·4), (1·6), (2·1) and (2·3), the equation (2·8) yields the result (2·6).

Theorem 2·2. Under the decompositions (2·1) and (2·3), the tensor fields ϕ_{jkh} and ϕ_{kh} behave like recurrence tensor fields with respect to the recurrence vector field v_l.

Proof. Differentiating (2·3) covariantly and using (1·7), we have

(2·9) $$v_l H^i_{jkh} = \dot{x}^i \phi_{jkh(l)}.$$

By virtue of decomposition (2·3), it becomes

(2·10) $$\phi_{jkh(l)} = v_l \phi_{jkh}.$$

Transvecting (2·10) $\dot{x}^j$, we have

(2·11) $$\phi_{kh(l)} = v_l \phi_{kh} ,$$

which proves the theorem.

Theorem 2·3. Under the decomposition (2·1), the recurrence vector v_m satisfies the relation

(2·12) $$\dot{\partial}_l v_m = \dot{\partial}_m v_l .$$

Proof. Differentiating (2·11) with respect to $\dot{x}^m$, we have

(2·13) $$\dot{\partial}_m \phi_{kh(l)} = \dot{\partial}_m v_l \phi_{kh} + v_l \dot{\partial}_m \phi_{kh} .$$

With the help of commutation formula (1·11), it becomes

(2·14) $$- \phi_{rh} G^r_{kml} - \phi_{kr} G^r_{hml} = \dot{\partial}_m v_l \phi_{kh} .$$

But G^i_{jkh} is symmetric with respect to the indices k and h and therefore by interchanging the indices l and m in (2·14), we get

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$(2\cdot15)$ $$(\partial_m\, v_l - \partial_l\, v_m)\, \phi_{kh} = 0.$$

From the assumption $\phi_{kh} \neq 0$, Theorem 2·3 is established.

Theorem 2·4. Under the decomposition (2·1) the relation

$(2\cdot16)$ $$v_m\, \phi_{kh} = v_k\, \phi_{mh}$$

in true.

Proof. Interchanging the indices m and k in (2·14) and subtracting the result obtained from it, we get

$(2\cdot17)$ $$- \phi_{kr}\, G^r_{hml} + \phi_{mr}\, G^r_{hkl} = \partial_m\, v_l\, \phi_{kh} - \partial_k\, v_l\, \phi_{mh}.$$

Multiplying (2·17) by $\dot{x}^l$, we obtain

$(2\cdot18)$ $$\partial_m\, v_l\, \phi_{kh}\, \dot{x}^l = \partial_k\, v_l\, \phi_{mh}\, \dot{x}^l.$$

With the help of $v_l\, \dot{x}^l = \rho$ (constant), we get the result (2·16).

Theorem 2·3. Under the decompositions (2·1) and (2·3), the tensor field ϕ_{jk} can be expressed as

$(2\cdot19)$ $$\phi_{lm} = - (v_{l(m)} - v_{m(l)}).$$

Proof. Differentiating (2·11) covariantly with respect to x^m and commuting the indices l and m in it, we have

$(2\cdot20)$ $$(v_{l(m)} - v_{m(l)})\, \phi_{kh} = \phi_{kh(l)(m)} - \phi_{kh(m)(l)}.$$

Using the commutation formula (1·9), we get

$(2\cdot21)$ $$(v_{l(m)} - v_{m(l)})\, \phi_{kh} = - \partial_r\, \phi_{kh}\, H^r_{lm} - \phi_{rh}\, H^r_{klm} - \phi_{kr}\, H^r_{hlm}\ .$$

By virtue of decomposition (2·1) and (2·3) and the assumption $\phi_{kh}\, \dot{x}^h = 0$, it becomes

$(2\cdot22)$ $$(v_{l(m)} - v_{m(l)})\, \phi_{kh} = - \phi_{kh}\, \phi_{lm}.$$

But $\phi_{kh} \neq 0$, hence Theorem 2·5 is proved.

3. Decomposition of projective curvature tensor and projective tensor fields.

Considering the decomposition of W^i_{jkh} in the form

$(3\cdot1)$ $$W^i_{jkh} = \dot{x}^i\, \psi_{jkh},$$

where ψ_{jkh} is non–zero homogeneous tensor field of degree zero in $\dot{x}^i$.

Transvecting (3·1) with respect to $\dot{x}^j$, we get

$(3\cdot2)$ $$W^i_{kh} = \dot{x}^i\, \psi_{kh},$$

where $$\dot{x}^j\, \psi_{jkh} = \psi_{kh}.$$

Thus we have the following theorems.

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Theorem 3·1. Under the decomposition (3·1), the identity satisfied by the projective curvature tensor field have the form

(3·3) $$\psi_{jkh} + \psi_{khj} + \psi_{hjk} = 0.$$

Proof. By virtue of decomposition (3·1), it is obvious from (1·16).

Theorem 3·2. Under the decompositions (3·1) and (3·2), the tensor fields ψ_{jkh} and ψ_{kh} are recurrence tensor fields with respect to the recurrence vector field v_l and their recurrence forms are

(3·4) $$\psi_{jkh(l)} = v_l \,\psi_{jkh}$$

and

(3·5) $$\psi_{kh(l)} = v_l \,\psi_{kh}$$

respectively.

Pr of. It follows the pattern of Theorem 2·2.

Theorem 3·3. Under the decomposition (3·1) the recurrence vector v_l satisfies the relation

(3·5) $$\partial_j \, v_m = \partial_m \, v_l.$$

Proof. Its method is similar to that of Theorem 2·3.

Theorem 3·4. Under the decomposition (3·1), the relation

(3·6) $$v_m \, \psi_{kh} = v_k \, \psi_{mh}$$

holds good.

Proof. It follows the method of Theorem 2·4.

Theorem 3·5. Under the decompositions (2·1) and (2·3), the tensor ψ_{jk} satisfies the relation

(3·7) $$\psi_{rh} \, \dot{x}^r = 0.$$

Proof. Differentiating (3·5) covariantly with respect to x^m and substracting the result obtained by interchanging the indices l and m in it, from it, we have

(3·8) $$(v_{l(m)} - v_{m(l)}) \, \psi_{kh} = \psi_{kh(l)(m)} - \psi_{kh(m)(l)}.$$

By using the commutation formula (1·9), it becomes

(3·9) $$(v_{l(m)} - v_{m(l)}) \, \psi_{kh} = - \partial_r \, \psi_{kh} \, H^r_{lm} - \psi_{rh} \, H^r_{klm} - \psi_{kr} \, H^r_{hlm} .$$

From decompositions (2·1) and (2·3), we get

(3·10) $$(v_{l(m)} - v_{m(l)}) \, \psi_{kh} = - \psi_{kh} \, \phi_{lm} - \psi_{rh} \, \dot{x}^r \, \phi_{klm} - \psi_{kr} \, \dot{x}^r \, \phi_{hlm}.$$

By virtue of (2·19), we have

(3·11) $$\psi_{rh} \, \dot{x}^r \, \phi_{klm} + \psi_{kr} \, \dot{x}^r \, \phi_{hlm} = 0.$$

Multiplylng (3·11) by $\dot{x}^k$ and because $\phi_{lm} \neq 0$, we get the result.

References

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On the formal solution of simultaneous Dual integral equations

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1. Introduction

Simultaneous dual integral equations arise in the formulation of great variety of mixed boundary value problems of semi–infinite domains involving more than one unknown function such as the problems in elastostatics, visco-elasticity and electrostatics. The object of the present paper is to obtain a formal solution of certain simultaneous dual integral equations involving general H–functions as kernels. The importance of the result established lies in the fact that the solution of simultaneous dual integral equations possessing general Meijer's G–functions and nearly all the special functions occurring in Applied Mathematics can be derived as particular cases from our result due to most general character of H–function.

We define the general H–function by means of the Mellin–Barnes integral [2, p. 49]

$$H^{m,n}_{p+n,\,q+m}\left(x \left|\begin{array}{c} a_i, A_i \\ b_j, B_j \end{array}\right.\right)$$

$$= H^{m,n}_{p+n,\,q+m}\left(x \left|\begin{array}{c} a_1, A_1, a_2, A_2, \ldots, a_{n+p}, A_{n+p} \\ b_1, B_1, b_2, B_2, \ldots, b_{m+q}, B_{m+q} \end{array}\right.\right) \qquad (1)$$

$$= \frac{1}{2\pi i}\int_c \frac{\prod\limits_{i=1}^{m}\Gamma(b_i+sB_i)\prod\limits_{i=1}^{n}\Gamma(a_i-sA_i)}{\prod\limits_{i=1}^{q}\Gamma(b_{m+i}-sB_{m+i})\ \prod\limits_{i=1}^{p}\Gamma(a_{n+i}+sA_{n+i})} \qquad (2)$$

where an empty product is to be interpreted as unity and the following simplified assumptions are made.

(i) a_i, A_i, b_j and B_j are all real for $i = 1, 2, \ldots, (p+n)$ and $j = 1, 2, \ldots, (q+m)$.

(ii) A_i's and B_j's are positive for $i = 1, 2, \ldots, (p+n)$ and $j = 1, 2, \ldots, (q+m)$.

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(*iii*) All the poles of the integrand in (2) are simple.

(*iv*) Let $s = \sigma + it$, σ and t being real; then the contour along which the integral of (2) is taken, is a straight line parallel to the imaginary axis in the complex S-plane whose equation is $\sigma = \sigma_0$, where σ_0 is a constant. The contour is such that all the poles of $\Gamma(b_j + s B_j)$ for $j = 1, 2, \ldots\ldots, m$ lies to the left and those of $\Gamma(a_i - s A_i)$ for $i = 1, 2, \ldots\ldots, n$ to the right of it.

(*v*)
$$\sum_{i=1}^{n} A_i + \sum_{i=1}^{m} B_i = \sum_{i=1}^{p} A_{n+i} + \sum_{i=1}^{q} B_{m+i}.$$

(*vi*)
$$\lambda = - \sum_{i=1}^{m} B_i + \sum_{i=1}^{n} A_i - \sum_{i=1}^{q} B_{m+i} + \sum_{i=1}^{p} A_{n+i},$$

(*vii*)
$$\mu = \sum_{i=1}^{n} a_i + \sum_{i=1}^{m} b_i - \sum_{i=1}^{p} a_{n+i} - \sum_{i=1}^{q} b_{m+i} + \tfrac{1}{2}(p + q - m - n).$$

(*viii*) $\quad x > 0.$

(*ix*) $\quad \lambda \sigma_0 + \mu + 1 < 0.$

It follows from the asymptotic representation of the gamma function [1, p. 47]

$$\lim_{|t| \to \infty} \left| \Gamma(\sigma + it) \right| \, |t|^{\frac{1}{2} - \sigma} \, \exp(\pi/2 |t|) = (2\pi)^{\frac{1}{2}},$$

on setting $s = \sigma_0 + it$, $x = Re^{i\phi}$ ($R > 0$, ϕ real) that the absolute value of the integrand of (2) is comparable with $(\rho/R)^{\sigma_0} |t|^{\lambda \sigma_0 + \mu} e^{\phi t}$

where $\quad \rho = \prod_{i=1}^{n} (A_i)^{A_i} \prod_{i=1}^{m} (B_i)^{-B_i} \prod_{i=1}^{p} (A_{n+i})^{A_{n+i}} \prod_{i=1}^{q} (B_{n+i})^{-B_{n+i}},$

when $|t|$ is large and therefore the integral (2) converges absolutely if the conditions (*viii*) and (*ix*) are satisfied.

The formal solution of the following simultaneous dual integral equations will be obtained here

$$\int_0^\infty H_{p+n,\ q+m}^{m,n} \left(x u \left| \begin{matrix} a_i, A_i \\ \ ^k \\ b_j, B_j \end{matrix} \right. \right) \sum_{h=1}^{n} a_{hk} f_h(u)\, du = \phi_k(x), \quad 0 < x < 1, \qquad (3)$$

$$\int_0^\infty H_{p+n,\ q+m}^{m,n} \left(x u \left| \begin{matrix} c_i, A_i, \ ^k \, c_{n+i}, A_{n+i} \\ \ ^k \\ d_j, B_j, d_{m+j}, B_{m+j} \end{matrix} \right. \right) \sum_{h=1}^{n} b_{hk} f_h(v)\, du = \psi_k(x),$$
$$\qquad\qquad\qquad\qquad\qquad x > 1;$$
$$k = 1, 2, \ldots, n, \qquad (4)$$

where a_{hk} and b_{hk} are constants, $\phi_k(x)$ and $\psi_k(x)$ are given and $f_h(x)$ are to be found. We assume that the H-function of (3) satisfies all the conditions given

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above with a_i replaced a_i^k for $i = 1, 2, \ldots, (n + p)$ and b_j replaced by b_j^k for $j = 1, 2, \ldots, (m + q)$. Similarly the H-function of (4) satisfies all the conditions given above with a_i replaced by c_i for $i = 1, 2, \ldots, n$ and a_{n+i} replaced by c_{n+i}^k for $i = 1, 2, \ldots m\, p$; also b_j replaced by d_j^k for $j = 1, 2, \ldots, m$ and b_{m+j} replaced by d_{m+j} for $j = 1, 2, \ldots, q$. We also assume that the common value of σ_0 can be found for both the H-functions.

The solution of the equation (3) and (4) is also obtained when a_{n+i}^k is replaced by a_{n+i}^k for $i = 1, 2, \ldots, p$ and b_i^k is replaced by b_i for $i = 1, 2, \ldots, m$; and c_i is replaced by c_i^k for $i = 1, 2, \ldots, n$ and d_{m+i} is relpaced by d_{m+i}^k for $i = 1, 2, \ldots, q$.

2. *The Mellin transform.* From (2), it follows that

$$
m\left[H\left(x \,\middle|\, \begin{matrix} a_i^k, A_i \\ b_j^k, B_j \end{matrix} \right) \right] = \frac{\prod\limits_{i=1}^{m} \Gamma(b_i^k + s B_i)\, \prod\limits_{i=1}^{n} \Gamma(a_i^k - s A_i)}{\prod\limits_{i=1}^{q} \Gamma(b_{m+i} - s B_{m+i})\, \prod\limits_{i=1}^{p} \Gamma(a_{n+i}^k + s A_{n+i})} \tag{5}
$$

and

$$
m\left[H\left(x \,\middle|\, \begin{matrix} c_i, A_i, c_{n+i}^k, A_{n+i} \\ d_j^k, B_j, d_{m+j}, B_{m+j} \end{matrix} \right) \right] = \frac{\prod\limits_{i=1}^{m} \Gamma(d_i^k + s B_i)\, \prod\limits_{i=1}^{n} \Gamma(c_i - s A_i)}{\prod\limits_{i=1}^{q} \Gamma(d_{m+i} - s B_{m+i})\, \prod\limits_{i=1}^{p} (c_{n+i}^k + s A_{n+i})} \tag{6}
$$

where m denotes the Mellin transform.

On writing $m[f_h(u)] = F_h(s)$ and applying Parseral theorem [3, eq. (12)] to (3) and (4), these equations may be rewriten in the forms

$$
\frac{1}{2\pi i} \int_c \frac{\prod\limits_{i=1}^{m} \Gamma(b_i^k + sB_i)\, \prod\limits_{i=1}^{n} \Gamma(a_i^k - s A_i)\, x^{-s}}{\prod\limits_{i=1}^{q} \Gamma(b_{m+i} - s B_{m+i})\, \prod\limits_{i=1}^{p} \Gamma(a_{n+i}^k + s A_{n+i})}
$$

$$
\sum\limits_{h=1}^{n} a_{hk}\, F_h(1 - s)\, ds = \phi_k(x),\ 0 < x < 1, \tag{7}
$$

$$
\frac{1}{2\pi i} \int_c \frac{\prod\limits_{i=1}^{m} \Gamma(d_i^k + s B_i)\, \prod\limits_{i=1}^{n} \Gamma(c_i - s A_i)\, x^{-s}}{\prod\limits_{i=1}^{q} \Gamma(d_{m+i} - s B_{m+i})\, \prod\limits_{i=1}^{p} \Gamma(c_{m+i}^k + s A_{n+i})}
$$

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$$\sum_{h=1}^{n} b_{hk}\, F_h(1-s)\, ds = \psi_k(x),\ x>1\ ;\ k=1,2,\ldots,n. \tag{8}$$

3. *The reduction of (7) and (8) to equations with a common kernel.* In this section we will transform the equations (7) and (8) into two others with the same kernel by the application of integration operators. Our procedure is to transform

$$\frac{\prod\limits_{i=1}^{n} \Gamma(a_i - s\,A_i)}{\prod\limits_{i=1}^{q} \Gamma(b_{m+i} - s\,B_{m+i})}\ \text{of (7) into}\ \frac{\prod\limits_{i=1}^{n} \Gamma(c_i - s\,A_i)}{\prod\limits_{i=1}^{q} \Gamma(d_{m+i} - s\,B_{m+i})} \tag{9}$$

of (8) and

$$\frac{\prod\limits_{i=1}^{m} \Gamma(d_i + s\,B_i)}{\prod\limits_{i=1}^{p} \Gamma(c_{n+i} + s\,A_{n+i})}\ \text{of (8) into}\ \frac{\prod\limits_{i=1}^{m} \Gamma(e_i + s\,B_i)}{\prod\limits_{i=1}^{p} \Gamma(\gamma_{n+i} + s\,A_{n+i})} \tag{10}$$

and then

$$\frac{\prod\limits_{i=1}^{m} \Gamma(e_i + s\,B_i)}{\prod\limits_{i=1}^{p} \Gamma(\gamma_{n+i} + s\,A_{n+i})}\ \text{into}\ \frac{\prod\limits_{i=1}^{m} \Gamma(b_i + s\,B_i)}{\prod\limits_{i=1}^{p} \Gamma(a_{n+i} + s\,A_{n+i})} \tag{11}$$

of (7).

For making these transformations we will make use of the fractional integration operators defined by Erdelyi [1, p. 220]

$$F[\alpha,\beta:\mu:w(x)] = \frac{\mu}{\Gamma(\alpha)}\, x^{-\mu\alpha+\mu-\beta-1} \int_0^x (x^\mu - v^\mu)^{\alpha-1}\, v^\beta w(v)\, dv, \tag{12}$$

$$R[\alpha,\beta:\mu:w(x)] = \frac{\mu}{\Gamma(\alpha)}\, x^\beta \int_x^\infty (v^\mu - x^\mu)^{\alpha-1}\, v^{-\beta-\mu\alpha+\mu-1}\, w(v)\, dv. \tag{13}$$

When $\mu = 1$, (12) and (13) reduce to Kober's operators [4, p. 194]. Recently Fox [3, p. 393] has shown that there is no essential difference between the two operators. Both the operators F and R exist provided $w(x)\, \epsilon\, L_p'\,(0,\infty)$, $p' \geqslant 1$, $\alpha > 0$, $\beta > (1 - 'p)/p'$ [1, § 2]. If in addition can be differentiated sufficiently often, then the operators F and R exist for negative as well as positive value of α [1, 2, p. 222].

For brevity, we write

$$F[(a_i - c_i),\, c_i\, A_i^{-1} - 1 : A_i^{-1} : w(x)] = F_i[w(x)], \tag{14}$$

$$F[\,(d_{m+j} - b_{m+j}),\, b_{m+j}\, B_{m+j}^{-1} - 1 : B_{m+j}^{-1} : w(x)\,] = F_j{}^*\,[w(x)\,], \tag{15}$$

$$R[\,(d_e - e_e),\ e_e \overset{-1}{B_e} : \overset{-1}{B_e} : w(x)\,] = Re\,[w(x)\,] \qquad (16)$$

$$R[\,(\gamma_{n+h} - \overset{k}{c_{n+h}}),\ \overset{k}{c_{n+h}}\ \overset{-1}{A_{n+h}} : \overset{-1}{A_{n+h}} : w(x)\,] = R_h^*\,[w(x)\,] \qquad (17)$$

$$R[\,(e_\lambda - \overset{k}{b_\lambda}),\ \overset{k}{b_\lambda}\ \overset{-1}{B_\lambda} : \overset{-1}{B} : w(x)] = {}^*R_\lambda\,[w(x)] \qquad (18)$$

$$R[\,(\overset{k}{a_{n+\nu}} - r_{n+\nu}),\ r_{n+\nu}\ \overset{-1}{A_{n+\nu}} : \overset{-1}{A_{n+\nu}} : w(x)\,] = {}^{**}R_\nu[w(x)\,] \qquad (19)$$

Now we proceed to make the first transformation.

In (7) replaced x by v, multiply by $v^{cn\,\varepsilon n-1}\quad (x^{\varepsilon n} - v^{\varepsilon n})^{a_n - c_n - 1}$

where $\varepsilon_n = 1/A_n$, integrate with respect to v from 0 to x, and apply the Beta function formula [3, eq. (16)], we obtain

$$\frac{1}{2\pi i} \int_c \frac{\overset{m}{\underset{i=1}{\Pi}}\ \Gamma(\overset{k}{b_i} + s B_i)\ \overset{n-1}{\underset{i=1}{\Pi}}\ \Gamma(\overset{k}{a_i} - s A_i)\ \Gamma(c_n - s A_n)}{\overset{q}{\underset{i=1}{\Pi}}\ \Gamma(\overset{k}{b_{m+i}} - s B_{m+i})\ \overset{p}{\underset{i=1}{\Pi}}\ \Gamma(\overset{k}{a_{n+i}} + s A_{n+i})}\ x^{-s}$$

$$\overset{n}{\underset{h=1}{\Sigma}}\ a_{kk}\ F_h(1 - s)\ ds = \frac{\varepsilon n\ x^{\overset{k}{\varepsilon n - a_n}\ \varepsilon_n}}{\Gamma(\overset{k}{a_n} - c_n)} \int_0^x\ v^{\varepsilon_n\ c_n - 1}$$

$$\left(x^{\varepsilon n} - v^{\varepsilon n} \right)^{\overset{k}{a_n} - c_n - 1} \phi_k(v)\ dv = F_{nk}\ \phi_k(x)\,],$$

$$0 < x < 1\ ;\ k = 1, 2, \ldots, n. \qquad (20)$$

On transforming (20) successively, for $i = (n-1),\ (n-2),\ —,\ 2,\ 1$ by the application of the operators F_i and by F_i^* for $i = q,\ q-1,\ \ldots,\ 2,\ 1$, we get finally

$$\frac{1}{2\pi i} \int_c \frac{\overset{m}{\underset{i=1}{\Pi}}\ \Gamma(\overset{k}{b_i} + s B_i)\ \overset{n}{\underset{i=1}{\Pi}}\ \Gamma(c_i - s A_i)}{\overset{q}{\underset{i=1}{\Pi}}\ \Gamma(d_{m+i} - s B_{m+i})\ \overset{p}{\underset{i=1}{\Pi}}\ \Gamma(\overset{k}{a_{n+i}} + s A_{n+i})}\ x^{-s}$$

$$\overset{n}{\underset{h=1}{\Sigma}}\ a_{hk}\ F_h(1 - s)\ ds = F_1^*\ [F_2^*\ldots F_q^*\ F_1\ldots F_n\ [\phi_k(x)]\ldots\,],$$

$$0 < x < 1\ ;\ k = 1, 2, \ldots, n. \qquad (21)$$

Similarly if we transform (8) by the application of the operators R_l and R_h^* respectively for $l = m,\ m-1,\ \ldots,\ 2,\ 1$ and $h = p,\ p = 1,\ \ldots,\ 2,\ 1$, we obtain

$$\frac{1}{2\pi i} \int_c \frac{\prod\limits_{i=1}^{m} \Gamma(e_i + s\,B_i) \prod\limits_{i=1}^{n} \Gamma'(c_i - s\,A_i)}{\prod\limits_{i=1}^{q} \Gamma(d_{m+i} - s\,B_{m+i}) \prod\limits_{i=1}^{p} \Gamma(\gamma_{n+i} + s\,A_{n+i})} x^{-s}$$

$$\sum_{h=1}^{n} a_{hk}\, F_h(1-s)\, ds = \sum_{h=1}^{n} c_{hk}\, R_1^*\, [R_2^* \ldots R_p^*\, R_1 \ldots R_m[\psi_h(x)] \ldots],$$

$$x > 1\,; k = 1, 2, \ldots, n, \tag{22}$$

where c_{hk} are the elements of the matrix $[a_{hk}]\,[b_{hk}]^{-1}$.

Again on applying the operators $*R_\lambda$ and $**R_\nu$ respectively for $\lambda = m, m-1, \ldots, 2, 1$ and $\nu = p, p-1, \ldots 2, 1$ to (22) it may be written in the form

$$\frac{1}{2\pi i} \int_c \frac{\prod\limits_{i=1}^{m} \Gamma(b_i^k + s\,B_i) \prod\limits_{i=1}^{n} \Gamma(c_i - s\,A_i)\, x^{-s}}{\prod\limits_{i=1}^{q} \Gamma(d_{m+i} - s\,B_{m+i}) \prod\limits_{i=1}^{p} \Gamma(a_{n+i}^k + s\,A_{n+i})}$$

$$\sum_{h=1}^{n} a_{hk}\, F_h(1-s)\, ds = **R_1\, [**R_2 \ldots **R_p\, *R_1 \ldots *R_m$$

$$\left[\sum_{h=1}^{n} c_{hk}\, R_1^*\, [R_2^* \ldots R_p^*\, R_1 \ldots R_m\, [\psi_h(x)] \ldots \right],$$

$$x > 1\,; k = 1, 2, \ldots, n. \tag{23}$$

Finally on writting

$$t_k(x) = \begin{cases} F_1^*\,[F_2^* \ldots F_q^*\, F_1 \ldots F_n\,[\phi_k(x)] \ldots],\ 0 < x < 1, \\ **R_1\,[**R_2 \ldots **R_p\, *R_1 \ldots *R_m \sum\limits_{h=1}^{n} c_{hk} \\ R_1^*\,[R_2^* \ldots R_p^*\, R_1 \ldots R_m\,[\psi_h(x)] \ldots],\ x > 1\,; \\ \qquad\qquad k = 1, 2, \ldots, n, \end{cases} \tag{24}$$

we have, from (21) and (23)

$$\frac{1}{2\pi i} \int_c \frac{\prod\limits_{i=1}^{m} \Gamma(b_i^k + s\,B_i) \prod\limits_{i=1}^{n} \Gamma'(c_i - s\,A_i)\, x^{-s}}{\prod\limits_{i=1}^{q} \Gamma(d_{m+i} - s\,B_{m+i}) \prod\limits_{i=1}^{p} \Gamma(a_{n+i}^k + s\,A_{n+i})}$$

$$\sum_{h=1}^{n} a_{hk}\, F_h(1-s)\, ds = t_k(x),\ k = 1, 2, \ldots, n. \tag{25}$$

4. *The formal solution of* (7) *and* (8) : (25) is the reduction of (7) and (8) to two equations with a common kernel. On treating the kernel of (25) as an un symmetrical Fourier kernel and following a procedure adopted by Fox (3, § 4] it is found that (25) can be written in the form

$$f_h(x) = \frac{1}{2\pi i} \sum_{k=1}^{n} d_{hk} \int_c \frac{\prod\limits_{i=1}^{q} \Gamma(d_{m+i} + B_{m+i} - sB_{m+i})}{\prod\limits_{i=1}^{m} \Gamma(b_i + B_i - s B_i)}$$

$$\frac{\prod\limits_{i=1}^{p} \Gamma(a_{n+i} + A_{n+i} - s A_{n+i})x^{-s}}{\prod\limits_{i=1}^{n} \Gamma(c_i - A_i + s A_i)} \ T_k(1 - s)\ ds,\ h = 1, 2, \dots, n, \qquad (26)$$

where $mt(x) = T_k(s)$ and d_{hk} are the elements of the matrix $[a_{hk}]^{-1}$.

This is the formal solution of (7) and (8) and many important properties of the function $f_h(x)$ can be obtained from it. On applying the Parseval theorem [3, eq. (12,] to (26) we obtain that

$$f_h(x) = \sum_{k=1}^{n} d_{hk} \int_0^\infty H^{q,p}_{p+n,\ q+m} \left(x\,u \left| \begin{array}{c} a_{n+i} + A_{n+i},\ A_{n+i},\ c_i - A_i,\ A_i \\ d_{m+j} - B_{m+j},\ B_{m+j},\ b_j + B_j,\ B_j \end{array} \right. \right)$$
$$t_k(u)\ du, \quad h = 1, 2, \dots, n, \qquad (27)$$

where $t(x)$ is given by (24). The H-function of (27) exists if $\lambda\sigma_0 + \mu' + \lambda + 1 < 0$, where μ' is obtained from μ by interchanging $a_{i,s}$ by $- c_i\text{'s},\ i' = 1, 2, \dots, n$; $b_{i,s}$ by $- b_{i,s}$ for $i = 1, 2, \dots, m$; a_{n+i} by $- a_{n+i},\ b_{m+i}$ by $- d_{m+i}$ for $i = 1, 2, \dots, q$ and $\frac{1}{2}(p + q - m - n)$ by $\frac{1}{2}(m + n - p - q)$. Since our method is formal it does not give the conditions of validity of the solution.

The solution (27) can be verified by reversing all the steps from (3), (4) to (27) by means of the operators F^{-1} and R^{-1} defined by Fox [3, p. 397]. The H-function has been used here as an unsymmetrical Fourier kernel and the conditions under which H-function form a pair of unsymmetrical kernel are given in [5, p. 357].

The foregoing procedure is a straight forward generalisation of the method developed by Saxena for dual integral equations [6].

5. In a similar manner the formal solution of simultaneous dual integral equations

$$\int_0^\infty H^{m,n}_{p+n,\,q+m} \left(xu \left| \begin{array}{c} a_i,\ A_i,\ a_{n+i},\ A_{n+i} \\ b_j,\ B_j,\ b_{m+j},\ B_{m+j} \end{array} \right. \right) \sum_{h=1}^{n} a_{hk} f_h(u)\ du = \phi_k(x),$$
$$0 < x < 1, \qquad (28)$$

$$\int_0^\infty H^{m,n}_{p+n,\,q+m}\left(xu \,\middle|\, \begin{matrix} \overset{k}{c_i}, A_i, \\ \overset{k}{d_j}, B_j \end{matrix}\right) \sum_{h=1}^n b_{hk} f_h(u)\, du = \psi_k(x)\,;\ x > 1\,;$$

is given by
$$k = 1, 2, \ldots n, \tag{29}$$

$$f_h(x) = \sum_{k=1}^n d_{hk} \int_0^\infty H^{q,\,p}_{p+n,\,q+m}\left(xu \,\middle|\, \begin{matrix} a_{n+i} + A_{n+i},\ A_{n+i},\ \overset{k}{c_i} - A_i,\ A_i \\ \overset{k}{d_{m+j}} - B_{m+j},\ B_{m+j},\ b_j + B_j,\ B_j \end{matrix}\right)$$
$$\cdot\, t_k(u)\, du, \qquad h = 1, 2, \ldots, n, \tag{30}$$

where d_{hk} are the elements of the matrix $[b_{hk}]^{-1}$ and

$$t_k'(x) = \begin{cases} \overset{\psi\psi}{F_1}\,[\overset{\psi\psi}{F_2} \ldots \overset{\psi}{F_q}\,\overset{\psi}{F_1} \ldots \overset{\psi}{F_n}\,[\sum_{h=1}^n c_{hk} \\ \overset{\psi}{F_1}\,[\overset{\psi}{F_2} \ldots \overset{\psi}{F_q}\,F_1' \ldots F_n'\,[\,\phi_h(x)\,]\ldots],\ 0 < x < 1, \\ R_1\,[R_2 \ldots R_p\,R_1' \ldots R_m'\,[\,\psi_k(x)\,]\ldots],\ x > 1 \end{cases} \tag{31}$$

where c_{hk} are the elements of the matrices $[b_{hk}]\,[a_{hk}]^{-1}$ and

$$F\,[\,(\overset{k}{a_i} - e_i),\ e_i\,A_i^{-1} - 1,\ A_i^{-1} : w(x)\,] = F_i'\,[w(x)], \tag{32}$$

$$F\,[\,(r_{m+j} - \overset{k}{b_{m+j}}),\ \overset{k}{b_{m+j}}\,B_{m+j}^{-1} - 1 : \overset{-1}{B_{m+j}} : w(x)\,] = \overset{\psi}{F_j}\,[w(x)\,], \tag{33}$$

$$F\,[\,(e_\tau - \overset{k}{c_\tau}),\ \overset{k}{c_\tau}\,\overset{-1}{A_\tau} - 1 : \overset{-1}{A_\tau} : w(x)\,] = \overset{\psi}{F}[w(x)\,], \tag{34}$$

$$F\,[\,(\overset{k}{d_{m+\gamma}} - r_{m+\gamma}),\ r_{m+\gamma}\,\overset{-1}{B_{m+\gamma}} - 1,\ \overset{-1}{B_{m+\gamma}} : w(x)\,] = \overset{\psi\psi}{F_\gamma}\,[w(x)\,], \tag{35}$$

$$R\,[(\overset{k}{d_l} - b_l),\ b_l\,\overset{-1}{B_l} : \overset{-1}{B_l} : w(x)\,] = R'_l\,[w(x)\,], \tag{36}$$

$$R\,[\,(\overset{k}{c_{m+h}} - a_{m+h}),\ a_{m+h}\,\overset{-1}{A_{m+h}} : \overset{-1}{A_{m+h}} : w(x)\,] = \overset{\psi}{R_h}\,[w(x)]. \tag{37}$$

6. Particular cases. (i) If we set $n = p = 0$, $q = m$, $B_{m+i} = B_i = a_i$, $\overset{k}{b_i} = \overset{k}{\alpha_i}$, $b_{m+i} = \beta_i$, $d_i = \lambda_i$, $d_{m+i} = \mu_i$ for $i = 1, 2, \ldots, m$, in (3) and (4) then the formal solution of simultaneous dual integral equations

$$\int_0^\infty H\left(xu \,\middle|\, \begin{matrix} \overset{k}{a_i}, a_i \\ \overset{k}{\beta_i}, a_i \end{matrix} : m\right) \sum_{h=1}^n a_{hk} f_h(u)\, du = \phi_k(x),\ 0 < x < 1, \tag{38}$$

$$\int_0^\infty H\left(xu \,\middle|\, \begin{matrix} \lambda_i, a_i \\ \mu_i, a_i \end{matrix} : m\right) \sum_{h=1}^n b_{hk} f_h(u)\, du = \psi_k(x),\ x > 1\,;\ k = 1, 2, \ldots, n, \tag{39}$$

is given by

$$f_h(x) = \sum_{k=1}^{n} d_{hk} \int_0^{\infty} H\left(xu \left| \begin{array}{c} \mu_i - a_i, a_i, \\ k \\ \alpha_i + a_i, a_i \end{array} \right. : m \right) t_k(u)\, du, \; h = 1, 2, \ldots, n, \qquad (40)$$

where

$$t_k(x) = \begin{cases} F_1{}^* \left[F_1{}^* \ldots F_m{}^* \left[\phi_k(x) \right] \ldots \right], 0 < x < 1, \\[2mm] *R_1 \left[*R_2 \ldots *R_m \left[\sum_{h=1}^{n} c_{hk} R_1 \left[R_2 \ldots R_m \left[\psi_k(x) \right] \ldots \right] \right], x > 1, \end{cases} \quad (41)$$

and the operators are also to be taken with the above changes.

Similarly if we set $n = \rho = 0$, $q = m$, $B_{m+i} = B_i = a_i$, $\overset{k}{b_i} = a_i$, $\overset{k}{b_{m+i}} = \overset{k}{\beta_i}$, $\overset{k}{d_i} = \overset{k}{\lambda_i}$, $d_{m+i} = \overset{k}{\mu_i}$ for $i = 1, 2, \ldots, m$, in (28) and (29) the formal solution of simultaneous dual integral equations

$$\int_0^{\infty} H\left(xu \left| \begin{array}{c} a_i, a_i \\ k \\ \beta_i, a_i \end{array} \right. : m \right) \sum_{h=1}^{n} a_{hk} f_h(u)\, du = \phi_k(x), \; 0 < x < 1, \qquad (42)$$

$$\int_0^{\infty} H\left(xu \left| \begin{array}{c} \overset{k}{\lambda_i}, a_i \\ k \\ \mu_i, a_i \end{array} \right. : m \right) \sum_{h=1}^{n} b_{hk} f_h(u)\, du = \psi_k(x), \; x > 1; \qquad (43)$$

is given by

$$f_h(x) = \sum_{k=1}^{n} d_{hk} \int_0^{\infty} H\left(xu \left| \begin{array}{c} \overset{k}{\mu_i} - a_i, a_i \\ \alpha_i + a_i, a_i \end{array} \right. : m \right) t_k(u)\, du, \; h = 1, 2, \ldots, n, \qquad (44)$$

where

$$t_k(x) = \begin{cases} {}^{\psi\psi}F_1 \left[{}^{\psi\psi}F_2 \ldots {}^{\psi\psi}F_m \right] \sum_{h=1}^{n} c_{hk} \overset{\psi}{F_1} \left[\overset{\psi}{F_2} \ldots \overset{\psi}{F_m} \left[\phi_h(x) \right] \ldots \right], \\ \qquad\qquad 0 < x < 1, \\[2mm] R_1' \left[R_2' \ldots R_m' \left[\psi_k(x) \right] \ldots \right], x > 1, \end{cases} \quad (45)$$

and the operators are to be taken with the above changes.

The dual integral equations corresponding to above simultaneous equations were recently studied by Fox [3].

(ii) If we set $A_i = B_j = 1$ for $i = 1, 2, \ldots, (p + n)$ and $j = 1, 2, \ldots, (q + m)$, H-functions then reduce to Meijer's G-function [1, p. 207] and the equations (3) and (4) reduce to the following simultaneous dual integral equations

$$\int_0^{\infty} G_{p+n, q+m}^{m, n} \left(xu \left| \begin{array}{ccc} \overset{k}{a_1}, \ldots, & \overset{k}{a_{n+p}} \\ \overset{k}{b_1}, \ldots, & \overset{k}{b_{m+q}} \end{array} \right. \right) \sum_{h=1}^{n} a_{hk} f_h(u)\, du = \phi_k(x), \\ \qquad\qquad 0 < x < 1, \qquad (46)$$

$$\int_0^\infty G_{p\cdot n,\ q+m}^{m,n} \left(xu \ \middle| \ \begin{matrix} & & \overset{k}{} & & \overset{k}{} \\ c_1,\ \ldots,\ c_n,\ c_{n+1},\ \ldots,\ c_{n+p} \\ \overset{k}{d_1},\ \ldots,\ d_m,\ d_{m+1},\ \vdots\ \ldots,\ d_{m+j} \end{matrix} \right)$$

$$\sum_{h=1}^n b_{hk}\, f_h(u)\ du\ =\ \psi_k(x),\ x>1\ ;\ k=1,2,\ldots,n, \qquad (47)$$

whose formal solution is given by

$$f_h(x) = \sum_{k=1}^n d_{hk}\ \int_0^\infty G_{p+n,\ q+m}^{q,\ p} \left(xu \ \middle| \ \begin{matrix} \overset{k}{-a_{n+1}},\ \ldots,\ \overset{k}{-a_{n+p}},\ -c_1,\ \ldots,\ -c_n \\ -d_{m+1},\ \ldots,\ -d_{m+j},\ \overset{k}{-b_1},\ \ldots,\ \overset{k}{-b_m} \end{matrix} \right)$$

$$t_k(u)\ du,\quad h=1,2,\ldots,u, \qquad (48)$$

where

$$t_k(x) = \begin{cases} F_1^*\,[F_2^*\ldots F_q^*\,F_1\ldots F_n\,[\phi_k(x)]\ldots],\ 0<x<1, \\[2mm] {}^{**}R_1\,[{}^{**}R_2\ldots{}^{**}R_p\,{}^*R_1\ldots{}^*R_m\,\left[\displaystyle\sum_{h=1}^n c_{hk}\,R_1^*\ldots R_p^*\,R_1\ldots R_m\,[\psi_\omega(x)]\ldots\right] \\[4mm] \qquad\qquad x>1\ ;\ h=1,2,\ldots,n, \end{cases} \qquad (49)$$

The operators are to be taken with $\overset{k}{a_i}$ replaced by $1-\overset{k}{a_i}$, $i=1,2,\ldots,n$; $\overset{k}{b_{m+i}}$ by $1-\overset{k}{b_{m+i}}$ $i=1,2,\ldots,q$; c_i by $1-c_i,\ i=1,2,\ldots n$; $d_{m+i},\ i=1,\ldots q$; $B_i=A_j=1$, for $i=1,2,\ldots,(q+m)$ and $j=1,2,\ldots,(p+n)$, i. e., the operators will be due to Kober in this case.

Similarly the formal solution of the simultaneous dual integral equations

$$\int_0^\infty G_{p+n,\ q+m}^{m,\ n} \left(xu \ \middle| \ \begin{matrix} \overset{k}{a_i},\ \ldots,\ a_n, & \overset{k}{a_{n+1}},\ \ldots,\ a_{n+p} \\ b_1\ \ldots,\ b_m, & \overset{k}{b_{m+1}},\ \ldots,\ \overset{h}{b_{m+q}} \end{matrix} \right) \sum_{h=1}^n a_{hk}\, f_h(u)\ du$$

$$=\phi_k(x),\ 0<x<1, \qquad (50)$$

$$\int_0^\infty G_{p+n,\ q+m}^{m,\ n} \left(xu \ \middle| \ \begin{matrix} \overset{k}{c_1},\ \ldots,\ \ldots,\ \overset{k}{c_{n+p}}, \\ \overset{k}{d_1},\ \ldots,\ \ldots,\ \overset{k}{d_{m+1}}, \end{matrix} \right)\ \cdot\ \sum_{h=1}^n b_{hk}\, f_h(u)\ du\ =\ \psi_k(x),\ x>1;$$

$$k=1,2,\ldots,n, \qquad (51)$$

is given by

$$f_h(x) = \sum_{k=1}^n d_{hk} \int_0^\infty G_{p+n,\ q+m}^{q,p} \left(xu \ \middle| \ \begin{matrix} -a_{n+1},\ \ldots,\ a_{n+p},\ \overset{k}{-c_1},\ \ldots,\ \overset{k}{-c_n} \\ \overset{k}{-d_{m+1}},\ \ldots,\ \overset{k}{d_{m+q}},\ -b_1,\ \ldots,\ -b_m \end{matrix} \right)$$

$$t_k(u)\ du,\quad h=1,2,\ldots,n, \qquad (52)$$

where

$$t_k(x) = \begin{cases} {}^{\psi\psi}F_1 \; [{}^{\psi\psi}F_2 \ldots {}^{\psi\psi}F_q \; {}^{\psi}F_1 \ldots {}^{\psi}F_n \; [\sum_{h=1}^{n} c_{hk} \, F_1{}^{\psi} \; [F_2{}^{\psi} \ldots F_q{}^{\psi} \\ \qquad . \; F_1' \ldots F_n' \; \lfloor \phi_h(x) \rfloor \ldots \rfloor , \; 0 < x < 1, \end{cases}$$

$$R_1{}^{\psi} \, [R_2{}^{\psi} \ldots R_p{}^{\psi} . R_1' \ldots , R_m' \, [\, \psi_k(x) \,] \ldots], \; x > 1, \; k = 1, 2, \ldots, n, \quad (53)$$

where the operators are to be taken with $\overset{k}{a_i}$ replaced by $1 - \overset{k}{a_i}$

$i = 1, 2, \ldots, n$; $\overset{k}{b}_{m+i}$ by $1 - \overset{h}{b}_{m+i}$, $i = 1, 2, \ldots, q$; $\overset{k}{c_i}$ by $1 - \overset{k}{c_i}$,

$i = 1, 2, \ldots, n$; $\overset{k}{d}_{m+i}$ by $1 - \overset{h}{d}_{m+i}$ $i = 1, 2, \ldots, q$; $B_i = A_j = 1$,

$i = 1, 2, \ldots, (q + m)$ and $j = 1, 2, \ldots, (p + n)$.

Acknowledgement

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On Steady Spatial Gas Flows

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Introduction

Smith (1963) in his ivestigation has shown that the normals to isobars, isovels, isentropics and isopynics, are coplanar for the reduced velocity vector field of an inviscid and thermally non-conducting steady gas flows, for the product equation of state. A similar result has been proved by Hassen and Martin[1] for plane steady gas flows. Purushotham[3] has solved the same problem for actual velocity vector field with any general equation of state. Later in 1965 Purushotham and Chari have proved that the normals to the isobars, isopynics, isovels, isotherms, and the equi-enthalpy lines are coplanar. Here in this paper we have proved that in addition to these normals, the normals to the local velocity constant surfaces and machvels are coplanar for any general equation of state.

Considering the Complex-Lammellar gas flows we have proved that the vorticity shall also lie in the plane in which the above normals lie. Also we have proved that the normal to the Beltrami surfaces and the streamlines shall be orthogonal to all the above normals.

§1. Basic Equations :

The basic equations governing an inviscid, thermally non-conducting with the two specific heats constant, steady Gas flows in the absence of external forces are given below in the usual notation[2] :

(1)
$$\operatorname{div}(\rho \vec{q}) = 0$$

(2)
$$\frac{1}{\rho} \nabla p + \operatorname{curl}\vec{q} \wedge \vec{q} + \tfrac{1}{2} \nabla q^{2l} = 0$$

(3)
$$\vec{q} \cdot \nabla S = 0$$

(4)
$$\rho = \rho, (p, S)$$

Using (2), (3) and (4) we have

(5)
$$q^2 + 2 h(p, S) = 2 h_0(p_0, S) = a^2$$

where the enthalpy $h(p, S)$ is defined by

(6)
$$h(p, S) = \int_0^p \frac{dp}{\rho(p, S)}$$

and $h_0(p_0, S)$, a^2 are constant along individual stream-line.

Also we have Crocco's vorticity equation and the velocity of the sound.

(7)
$$\vec{q} \wedge \text{curl } \vec{q} = \nabla\left\{ \frac{q^2}{2} + h(p, S) \right\} - T \nabla S$$

(8)
$$c^2 = \left(\frac{\partial p}{\partial \rho} \right)_S$$

Introducing the Bernoulli's surface, the equation (7) can be written as

(9)
$$\nabla B = T \nabla S + \vec{q} \wedge \text{curl } \vec{q}$$

§2.   In this section we shall prove that the normals to the isobars, isopycnics, isentropics isotherms, isovels, Bernoulli's surfaces, isovels of sound, vortex lines, Machvels and equi-enthalpy lines are coplanar.   Also the normals to the Beltrami surfaces are orthogonal to these normals.

Operating grad on (4) and taking the vector product by ∇p we have

(10)
$$\nabla \rho \wedge \nabla p = \frac{\partial \rho}{\partial S} \nabla S \wedge \nabla p$$

Operating grad on (5) and forming the vector product by ∇S we get

(11)
$$q (\nabla q \wedge \nabla S) + \frac{\partial h}{\partial p} \nabla \rho \wedge \nabla p = a \nabla a \wedge \nabla S$$

Forming the vector product of (10) and (11) we obtain

(12) $$q(\nabla q \wedge \nabla S) \wedge (\nabla \rho \wedge \nabla p) = -\frac{\partial \rho}{\partial S} (\nabla S \wedge \nabla p) \wedge (a \nabla a \wedge \nabla S)$$

For homenergic flows this simplifies to

(13)
$$(\nabla q \wedge \nabla S) \wedge (\nabla \rho \wedge \nabla p) = 0$$

from this we have following result :

The normals to the isobars, isovels, isopycnics and isentropics are coplanar, for homenergic steady gas flows.

Also operating curl on (2), (9) and forming the vector product we get

(14)
$$(\nabla \rho \wedge \nabla p) \wedge (\nabla S \wedge \nabla T) = 0$$

Using (2) and (7) we obtain

(15)
$$\frac{1}{\rho} \nabla p = \nabla h - T \nabla S$$

eliminating the pressure from (15) we obtain

(16)
$$(\nabla \rho \wedge \nabla h) \wedge (\nabla T \wedge \nabla S) = 0$$

Forming the vector product of (9) by ∇T we have

(17)
$$\nabla B \wedge \nabla T = T \nabla S \wedge \nabla T + (\vec{q} \wedge \text{curl} \vec{q}) \wedge \nabla T$$

multiplying (17) vectorically by $\nabla \rho \wedge \nabla p$ and using (13), (14) and (15) we get

(18)
$$(\nabla B \wedge \nabla T) \wedge (\nabla \rho \wedge \nabla p) = 0$$

Operating grad on (8) and forming vector product by ∇S we get

$$(19) \qquad \frac{\partial^2 p}{\partial \rho^2} (\nabla \rho \wedge \nabla S) = \nabla c^2 \wedge \nabla S$$

Forming the vector product by $\nabla q \wedge \nabla p$ and using (13) we have

$$(20) \qquad (\nabla c \wedge \nabla S) \wedge (\nabla q \wedge \nabla p) = 0$$

Using "Mc' for 'q' in (20) we get

$$(21) \qquad (\nabla M \wedge \nabla S) \wedge (\nabla p \wedge \nabla \rho) = 0$$

From the conditions (14), (16), (18), (20) and (21) we have the result:

For homenergic, steady spatial gas flows, the normals to isobars, isopycnics, isentropics, isovels, machvels, soundvels, Bernoulli's surfaces equi-enthalpy lines and isotherms are coplanar.

This generalises the previous theorems established by Hassen and Martin[1] Smith[5] Purushotham[3] and Chari and Purushotham[4].

(B) Complex-Lammellar Flow

A complex Lammellar flow is one in which the streamlines are normal to a one parameter family of surfaces, *i.e.*, it is a field of the type[7].

$$(22) \qquad \vec{q} = \alpha \operatorname{grad} \phi$$

were α and ϕ are scalar point functions.

The surfaces obtained by $\phi =$ constant is defined to be the Beltrami surfaces[6].

Using (22) in (9) and forming the vector product by ∇B we have

$$(23) \qquad \nabla B \wedge \nabla S = -\alpha (\nabla S. \operatorname{curl} \vec{q}) \nabla \phi$$

Taking the vector product of (23) by

$$(25) \qquad \nabla \phi . \nabla B = 0$$

We conclude from (24) that the normals to the Beltrami surfaces are orthogonal to the above nine normals.

The condition that the streamlines may be normals to a single parametric family of surfaces is

$$(25) \qquad \nabla \phi . \operatorname{curl} \vec{q} = 0$$

This shows that the vortexline for complex-lamellar gas flows lie in the plane of above normals.

It is evident from the relation (22) that the streamlines are also orthogonal to the above normals and the vortexlines.

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On certain inequalities satisfied by the smallest positive Zeros of a transcendental function and its higher derivatives.

By

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Abstract

Previously we have come across the inequalities satisfied by the smallest positive zeros j_ν, j_ν' and j_ν'', of the functions $J_\nu(z), J_\nu'(z)$ and $J_\nu''(z)$ respectively, under the restriction that $\nu + 1 > 0$.

The object of the present paper is to extend some of these results to the inequalities satisfied by the smallest positive zeros $j_{\nu,h,1}$, $j'_{\nu,h,1}$ and $j''_{\nu,h,1}$, of the respective functions $\psi_\nu^h(z)$, $\psi_\nu'^h(z)$ and $\psi_\nu''^h(z)$; where $\psi_\nu^h(z)$ is defined by the relation

$$\psi_\nu^h(z) \equiv \frac{h}{z} J_\nu(z) + J_\nu'(z),$$

subject to the restrictions $h \geqslant \nu > 1$, with the addition that $\nu^2 > 2h - 1$ and $\nu^2 > h^2 - \frac{h}{2} + \frac{1}{4}$ in certain cases related to $j_{\nu,h,1}, j'_{\nu,h,1}$ and $j''_{\nu,h,1}$.

1. Introduction

Previously[1] we have come across the results concerning the inequalities satisfied by the smallest positive zeros j_ν, j_ν' and j_ν'' of the function $J_\nu(z), J_\nu'(z)$ and $J_\nu''(z)$ respectively, under the assumption that $\nu + 1 > 0$.

The present paper consists of certain extensions of the abovementioned results to the smallest positive zeros $j_{\nu,h,1}$, $j'_{\nu,h,1}$ and $j''_{\nu,h,1}$ of the functions $\psi_\nu^h(z)$, $\psi_\nu'^h(z)$ and $\psi_\nu''^h(z)$ respectively, where $\psi_\nu^h(z)$ is defined by the relation

$$\psi_\nu^h(z) \equiv \frac{h}{z} J_\nu(z) + J_\nu'(z), \tag{1.1}$$

where h is a real constant.

Obviously, by virtue of the result[2].

$$J_{\nu-1}(z) = \frac{\nu}{z} J_\nu(z) + J_\nu'(z), \tag{1.2}$$

(1.1) assumes the form

$$\psi_\nu^h(z) \equiv \frac{(h-\nu)}{z} J_\nu(z) + J_{\nu-1}(z), \tag{1.3}$$

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which may be reduced to the form

$$\psi_\nu{}^\nu(z) = J_{\nu-1}(z),\tag{1.4}$$

for $h = \nu$. Thus, the results of $\psi_\nu{}^h(z)$ may easily be checked with those of $J_{\nu-1}(z)$ under the special case when $h = \nu$.

In one of our previous publications[3] we have discussed that the zeros of the function $z\,\psi_\nu{}^h(z)$ are all real when $h + \nu \geqslant 0$ and there exists two imaginary zeros $\pm i\,\lambda_0$ when $h + \nu < 0$ and the smallest positive zero $j_{\nu,h,1}$ satisfies the inequality

$$\left[\sigma_{\nu,h,1}^{(i)} \right]^{-1/2i} < j_{\nu,h,1} < \left[\sigma_{\nu,h,1}^{(i)} \middle/ \sigma_{\nu,h,1}^{(i+1)} \right]^{\frac{1}{2}},\tag{1.5}$$

where $\qquad \sigma_{\nu,h,1}^{(i)} = \sum_{m=1}^{\infty} 1 \middle/ j_{\nu,h,m}^{2i} \qquad$ ($i \equiv$ a positive integer).

In the present paper, an attempt has been made to establish certain inequalities satisfied by $j_{\nu,h,1}$, $j'_{\nu,h,1}$ and $j''_{\nu,h,1}$ which involve numbers depending purely on h and ν subject to the restrictions $h \geqslant \nu > 1$ with the addition that $\nu^2 + 1 > 2h$ and $\nu^2 > h^2 - \frac{h}{2} + \frac{1}{2}$ in certain results concerning $j_{\nu,h,1}$, $j'_{\nu,h,1}$ and $j''_{\nu,h,1}$. The condition $\nu^2 > \text{Max}\,[2h-1,\ h^2 - \frac{h}{2} + \frac{1}{2}]$ has been further utilized to determine the range of values of h and ν for which $\psi_{\nu+m}^{h}\,(j)_{\nu,h,1} \geqslant 0$ for positive integral values of m; and the result has been checked for the special values $m = 1, 2, 3$ and 4; as furnished by the Table I under the section 13.

To establish the required inequalities, however, we have felt the necessity of determining certain recurrence relations and the differential equation for $\psi_\nu{}^h(z)$, as incorporated in the beginning of the present paper. The validity of these inequalities has been checked numerically for certain special values of h and ν; as furnished by the sections (10–16).

2. Differential equation for $\psi_\nu{}^h(z)$:

Obviously, $\qquad z\,F_\nu{}'^h(z) = z^2 J_\nu{}''(z) + (h + 1)\,z\,J_\nu{}'(z),\tag{2.1}$

where $\qquad F_\nu{}^h(z) = z\,\psi_\nu{}^h(z).$

By virtue of the differential equation

$$\left(z\,\frac{d}{dz} \right)^2 J_\nu(z) = (\nu^2 - z^2)\,J_\nu(z)\tag{2.2}$$

(2.1) may be expressed in the form

$$z\,F_\nu{}'^h(z) = (\nu^2 - h^2 - z^2)\,J_\nu(z) + h\,F_\nu{}^h(z).\tag{2.3}$$

Again, differentiating with respect to 'z', we obtain

$$\left(z\,\frac{d}{dz} \right)^2 F_\nu{}^h(z) = (\nu^2 - z^2)\,F_\nu{}^h(z) - 2\,z^2\,J_\nu(z)\tag{2.4}$$

which may easily be put into the form

$$\left(z\,\frac{d}{dz} \right)^2 \psi_\nu{}^h(z) = \{\,\nu^2 + 1 - 2h - z^2\,\}\,\psi_\nu{}^h(z) + \frac{2(h^2 - \nu^2)}{z}\,J_\nu(z),\tag{2.5}$$

which is the required differential equation for $\psi_\nu{}^h(z)$.

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2a. Special cases :

In particular, for $h = \nu$, (2·5) reduces to (2·2) with ν replaced by $\nu-1$.

3. *Recurrence relations for $\psi_\nu{}^h(z)$*

Relation I.
$$\psi_{\nu+1}^h(z) = \frac{(\nu-1)}{z}\,\psi_\nu{}^h(z) - \psi_\nu{}'^h(z) - \frac{J_{\nu+1}(z)}{z} \qquad (3·1)$$

To prove this, we make use of the results
$$z\,\psi_\nu{}^h(z) = (h+\nu)\,J_\nu(z) - z\,J_{\nu+1}(z)^{(3)} \qquad (3·2)$$
and
$$J_{\nu+1}(z) = \frac{\nu}{z}\,J_\nu(z) - J_\nu{}'(z),^{(2)} \qquad (3·3)$$

which may easily be combined into the form (3·1).

Relation II.
$$\psi_{\nu-1}^h(z) = \frac{(\nu+1)}{z}\,\psi_\nu{}^h(z) + \psi_\nu{}'^h(z) - \frac{J_{\nu-1}(z)}{z} \qquad (3·4)$$

which follows evidently by virtue of (1·2) and (1·3).

Relation III.
$$\frac{(h^2 - \nu^2 + z^2)}{zh}\,J_\nu{}'(z) = \frac{z}{h}\left\{1 - \frac{\nu^2 - h}{z^2}\right\}\psi_\nu{}^h(z) + \psi_\nu{}'^h(z). \qquad (3·5)$$

By virtue of (1·1), we find that
$$\psi_\nu{}'^h(z) = \frac{d}{dz}\left\{\frac{h}{z}J_\nu(z) + J_\nu{}'(z)\right\} = J_\nu{}''(z) + \frac{h}{z}J_\nu{}'(z) - \frac{h}{z^2}J_\nu(z), \qquad (3·6)$$

which may be expressed in the form
$$\psi_\nu{}'^h(z) = \left\{\frac{\nu^2 - h}{z^2} - 1\right\}J_\nu(z) + \frac{h-1}{z}J_\nu{}'(z) \qquad (3·7)$$

by virtue of (2·2).

Combining (1·1) and (3·7), we arrive at (3·5).

Relation IV.
$$\psi_{\nu+2}^h(z) + \frac{2}{z}J_{\nu+2}(z) + \frac{2(h^2-\nu^2)}{hz^2}J_\nu{}'(z) = \left\{2\nu^2\left(\frac{h-1}{h}\right) - z^2\right\}\frac{\psi_\nu{}^h(z)}{z^2} - \frac{2\nu}{z}\psi_\nu{}'^h(z). \qquad (3·8)$$

Using (3·1) with ν replaced by $\nu + 1$ and expressing $\psi_{\nu+1}^h(z)$ in terms of $\psi_\nu{}^h(z)$ and $\psi_\nu{}'^h(z)$, we find

$$\psi_{\nu+2}^h(z) = \frac{\nu^2-1}{z^2}\psi_\nu{}^h(z) - \frac{2\nu-1}{z}\psi_\nu{}'^h(z) + \psi_\nu{}''^h(z) - \frac{1}{z}\left\{\frac{\nu+1}{z}J_{\nu+1}(z) - J'_{\nu+1}(z)\right\} - \frac{J_{\nu+2}(z)}{z} \qquad (3·9)$$

Again, using (2·5) and (3·3), (3·9) may be expressed in the form

$$\psi_{\nu+2}^h(z) = \frac{2(\nu^2-h)}{z^2}\psi_\nu{}^h(z) - \frac{2\nu}{z}\psi_\nu{}'^h(z) - \frac{2}{z}J_{\nu+2}(z) + \frac{2(h^2-\nu^2)}{z^3}J_\nu(z), \qquad (3·10)$$

which assumes (3·8) by virtue of (1·1).

Relation V.
$$\frac{(h-\nu)}{hz}\left\{hJ'_{\nu+2}(z) + J_\nu(z)\right\} + J'_{\nu+1}(z) + \frac{4(\nu+1)(h-\nu)}{z^3}J_{\nu+1}(z)$$

$$= \left\{ \frac{2\nu(\nu+1)}{z^2} - 1 \right\} \psi_\nu'^h(z) - \frac{2\nu}{z}\left\{ \frac{\nu^2-1}{z^2} - \frac{h-\nu+2\nu h}{2\nu h} \right\} \psi_\nu^h(z). \qquad (3\cdot11)$$

Using (1·3) with ν replaced by $\nu+2$, and combining it with (3·3) and (3·8), we get

$$\frac{h-\nu}{z} J_{\nu+2}(z) + J_{\nu+1}(z) + \frac{(h^2-\nu^2)}{hz^2}\left\{ \frac{\nu}{z} J_\nu(z) - J_{\nu+1}(z) \right\}$$

$$= \left\{ 2\nu^2 \frac{(h-1)}{h} - z^2 \right\} \frac{\psi_\nu^h(z)}{z^2} - \frac{2\nu}{z} \psi_\nu'^h(z), \qquad (3\cdot12)$$

which on using (3·2), assumes the form

$$\frac{h-\nu}{z} J_{\nu+2}(z) + \left\{ 1 - \frac{(2h-\nu)}{z^2} \right\} J_{\nu+1}(z) = \left\{ \frac{2\nu^2-2\nu}{z^2} - 1 \right\} \psi_\nu^h(z) - \frac{2\nu}{z} \psi_\nu'^h(z). \quad (3\cdot13)$$

Differentiating with respect to 'z', we get

$$\frac{h-\nu}{z} J'_{\nu+2}(z) - \frac{h-\nu}{z^2} J_{\nu+2}(z) + \left\{ 1 - \frac{2(h-\nu)}{z^2} \right\} J'_{\nu+1}(z) + \frac{4(h-\nu)}{z^3} J_{\nu+1}(z)$$

$$= \left\{ \frac{2\nu^2}{z^2} - 1 \right\} \psi_\nu'^h(z) - \frac{4\nu(\nu-1)}{z^3} \psi_\nu^h(z) - \frac{2\nu}{z} \psi_\nu''^h(z), \qquad (3\cdot14)$$

which may be written in the form

$$\frac{(h-\nu)}{z} J'_{\nu+2}(z) - \frac{(h-\nu)}{z^2} J_{\nu+2}(z) + \left\{ 1 - \frac{2(h-\nu)}{z^2} \right\} J'_{\nu+1}(z) + \frac{4(h-\nu)}{z^3} J_{\nu+1}(z)$$

$$= \left\{ \frac{2\nu(\nu+1)}{z^2} - 1 \right\} \psi_\nu'^h(z) - \frac{2\nu}{z}\left\{ \frac{\nu^2-1}{z^2} - 1 \right\} \psi_\nu^h(z) - \frac{4\nu(h-\nu)}{z^3} J_{\nu+1}(z), \quad (3\cdot15)$$

by virtue of (2·5).

Using the results

$$2 J'_{\nu+1}(z) = J_\nu(z) - J_{\nu+2}(z), \tag{2}$$

and (1·1) ; (3·15) may easily be cast into (3·11).

Relation VI.

$$\frac{1}{z} \psi_{\nu+1}^h(z) + \frac{1}{z^2} J_{\nu+1}(z) + \frac{2(h^2-\nu^2)}{z^3} J_\nu(z)$$

$$= \left\{ 1 - \frac{\nu^2+2-\nu-2h}{z^2} \right\} \psi_\nu^h(z) + \psi_\nu''^h(z), \qquad (3\cdot16)$$

which follows obviously by combining (3·1) and (2·5).

Relation VII.

$$\frac{(h-\nu)}{z} J'_{\nu+1}(z) + \left\{ 1 + \frac{(2\nu-1)(h^2-\nu^2)}{hz^2} \right\} J'_\nu(z)$$

$$= (\nu-1) \left\{ \nu^2 - 2\nu + \frac{\nu(2\nu-1)(h-\nu)}{h(\nu-1)} - z^2 \right\} \frac{\psi_\nu^h(z)}{z^2} - \nu \psi_\nu''^h(z) \qquad (3\cdot17)$$

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Using (1·3) with ν replaced by $\nu + 1$: and combining it with (3·1), we obtain

$$\frac{h-\nu}{z} J_{\nu+1}(z) + J_\nu(z) = \frac{(\nu-1)}{z} \psi_\nu{}^h(z) - \psi_\nu{}'^h(z).$$ (3·18)

Differentiating with respect to 'z' with the aid of (2·5), (3·18) may be written in the form

$$\frac{(h-\nu)}{z} J'_{\nu+1}(z) - \frac{h-\nu}{z^2} J_{\nu+1}(z) + J'_\nu(z)$$

$$= (\nu-1)\{\nu^2 - 2h - z^2\}\frac{\psi_\nu{}^h(z)}{z^2} - \nu\,\psi_\nu{}''^h(z) + \frac{2(\nu-1)(h^2-\nu^2)}{z^3} J_\nu(z),$$ (3·19)

which further reduces to the form

$$\frac{(h-\nu)}{z} J'_{\nu+1}(z) + J'_\nu(z) - \frac{(2\nu-1)(h-\nu)}{z^2} J_{\nu+1}(z)$$

$$= (\nu-1)\{\nu(\nu-2)-z^2\}\frac{\psi_\nu{}^h(z)}{z^2} - \psi_\nu{}''^h(z)$$ (3·20)

by virtue of (3·2).

Replacing $J_{\nu+1}(z)$ by $\frac{\nu}{h}\psi_\nu{}^h(z) - \frac{h+\nu}{h} J_\nu{}'(z)$, as obtained by combining (1·1) and (3·3). (3·20) may easily be cast into (3·17).

3a. *Special cases* :

In particular for $h = \nu$; (3·1), (3·4), (3·5), (3·8), (3·12), (3·16) and (3·17), respectively reduce to the forms

I. $\qquad J_\nu(z) = \frac{\nu-1}{z} J_{\nu-1}(z) - J'_{\nu-1}(z)$ (3a.1)

II. $\qquad J_{\nu-2}(z) = \frac{\nu-1}{z} J_{\nu-1}(z) + J'_{\nu-1}(z)$ (3a.2)

III. $\qquad J_\nu{}'(z) = \left\{1 - \frac{\nu^2-\nu}{z^2}\right\} J_{\nu-1}(z) + \frac{\nu}{z} J'_{\nu-1}(z)$ (3a.3)

IV. $\qquad J_{\nu+1}(z) = \left\{\frac{2\nu(\nu-1)}{z^2} - 1\right\} J_{\nu-1}(z) - \frac{2\nu}{z} J'_{\nu-1}(z)$ (3a.4)

V. $\qquad J'_{\nu+1}(z) = \left\{\frac{2\nu(\nu+1)}{z^2} - 1\right\} J'_{\nu-1}(z) - \frac{2\nu(\nu^2-1)}{z^3} J_{\nu-1}(z)$ (3a.5)

VI. $\qquad \frac{J_\nu(z)}{z} = \left\{1 - \frac{(\nu-1)(\nu-2)}{z^2}\right\} J_{\nu-1}(z) + J''_{\nu-1}(z)$ (3a.6)

VII. $\qquad J'_\nu(z) = (\nu-1)\left\{\frac{\nu^2-2\nu}{z^2} - 1\right\} J_{\nu-1}(z) - \nu\,J''_{\nu-1}(z)$ (3a.7)

which have been established previously[2].

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4. *Criterion for the existence of $j_{\nu,h,1}$ and $j'_{\nu,h,1}$*

Combining (1·3) and (2·5), we get

$$z \frac{d}{dz} [z \, \psi_\nu'^h(z)] = (h-\nu) \{ (\nu+1)^2 - z^2 \} \frac{J_\nu(z)}{z} + \{(\nu-1)^2 - 2(h-\nu) - z^2\} J_{\nu-1}(z) \quad (4\cdot1)$$

In accordance with the result

$$J_\nu(z) > 0, \text{ for } 0 < z \leqslant \nu, {}^{(1)} \qquad (4\cdot2)$$

it follows that the right hand side of (4·1) is a positive quantity in the range $0 < z \leqslant \sqrt{\nu^2+1-2h}$, subject to the restrictions $h \geqslant \nu > 1$.

Therefore $\psi_\nu^h(z)$ and $z \, \psi_\nu'^h(z)$ are positive increasing functions in the range $0 < z \leqslant \sqrt{\nu^2+1-2h}$, for $h \geqslant \nu > 1$.

On the other hand, using (3·2), (2·5) may be expressed as

$$z \frac{d}{dz} [z \, \psi_\nu'^h(z)] = (h+\nu) \{ (\nu-1)^2 - z^2 \} \frac{J_\nu(z)}{z} + \{z^2 - (\nu^2+1-2h)\} J_{\nu+1}(z), \qquad (4\cdot3)$$

which furnishes the idea that $\psi_\nu^h(z)$ and $z \, \psi_\nu'^h(z)$ are positive increasing functions in either of the ranges $\sqrt{\nu^2+1-2h} \leqslant z \leqslant \nu-1$ or $0 < z \leqslant \nu-1$, provided that $h \geqslant \nu > 1$.

In addition to this, it may be observed, by considering (1·3), that $\psi_\nu^h(z)$ remains positive in the range $\nu - 1 \leqslant z \leqslant j_{\nu-1}$, under the restrictions $h \geqslant \nu$.

It is, however, interesting to note that no real zero of $z \frac{d}{dz} [z \, \psi_\nu'^h(z)]$ exists in the range defined by $z < \sqrt{\nu^2+1-2h}$ for $h > \nu$.

For, combining (2·5) and (1·1), we get

$$z^2 \frac{d}{dz} [z \, \psi_\nu'^h(z)] = \{\nu^2 + 1 - 2h - z^2\} [z J_\nu'(z) + f(z) J_\nu(z)], \qquad (4\cdot4)$$

where

$$f(z) = \frac{h(\nu^2 + 1) - 2\nu^2 - hz^2}{\nu^2 + 1 - 2h} \sum_{m=0}^{\infty} (z^2/\nu^2 + 1 - 2h)^m, \qquad (4\cdot5)$$

with the addition that $|z| < \sqrt{\nu^2+1-2h}$.

Again, assuming 'α' to be a complex zero of

$$F_\nu(z) \equiv z J_\nu'(z) + f(z) J_\nu(z) \qquad (4\cdot6)$$

and α_0 its complex conjugate, we arrive at the result

$$\int_0^1 t J_\nu(\alpha t) \, J_\nu(\alpha_0 t) \, dt = \frac{J_\nu(\alpha) J_\nu(\alpha_0)}{\alpha^2 - \alpha_0^2} [f(\alpha) - f(\alpha_0)], \qquad (4\cdot7)$$

by virtue of Lommel's integral result[4].

Replacing α by $\rho e^{i\phi}$, (4·7) may easily be cast into the form

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$$\int_0^1 t\,(P^2 + Q^2)\,dt \;=\; \frac{2(R^2+S^2)\,(h^2-\nu^2)}{(\nu^2+1-2h)\,\rho^2\,\sin 2\phi} \sum_{m=0}^{\infty} (\rho^2/\nu^2+1-2h)^m \sin 2m\phi, \qquad (48)$$

where P, Q, R and S are real numbers.

Again, by virtue of the condition $\rho < \sqrt{\nu^2+1-2h}$, it can be proved (by trigonometry) that

$$\sum_{m=0}^{\infty} (\rho^2/\nu^2+1-2h)^m \sin 2m\phi \;=\; \frac{x \sin 2\phi}{(x-\cos 2\phi)^2+\sin^2 2\phi}, \qquad (4\cdot9)$$

where $x = \rho^2/\nu^2+1-2h$ and hence (4·8) assumes the form

$$\int_0^1 t\,(P^2+Q^2)\,dt \;=\; \frac{2(R^2+S^2)\,(h^2-\nu^2)}{(\nu^2+1-2h)^2\,\{(x-\cos 2\phi)^2+\sin^2 2\phi\}}, \qquad (4\cdot10)$$

which leads to the fact that so long as $h > \nu$, there are no real zeros of $F_\nu(z)$ in the range defined by $|z| < \sqrt{2^2+1-2h}$ and complex zeros might exist in this range.

In particular for $h = \nu$, (4·10) reduces to the form

$$\int_0^1 t\,(P^2+Q^2)\,dt = 0, \qquad (4\cdot11)$$

which is a contradiction, and hence there are no complex zeros of $J_{\nu-1}(z)$ in the range defined by $|z| < \nu-1$.

4a. Conclusion : It may be concluded that, so long as $\nu^2+1 > 2h$ and $h \geqslant \nu > 1$; $j'_{\nu,h,1}$ and $j_{\nu,h,1}$ will satisfy the inequalities

$$j'_{\nu,h,1} > \sqrt{\nu^2+1-2h} \qquad (4a.1)$$

and

$$j_{\nu\,h,1} > \sqrt{\nu^2+1-2h} \qquad (4a.2)$$

with the addition that

$$j'_{\nu,h,1} > \nu-1 \qquad (4a.3)$$

and

$$j_{\nu,h,1} \geqslant j_{\nu-1}, \qquad (4a.4)$$

subject to the restrictions $h \geqslant \nu > 1$ only.

4b. Special case : In particular, for $h = \nu$, the inequalities (4·12 − 4·15) reduce to the forms :

$$j''_{\nu-1} > \nu-1 \qquad (4b.1)$$

and

$$j_{\nu-1} > \nu-1, \qquad (4b.2)$$

where $\nu > 1$; as established previously[1].

4c. Relations involving $j'_{\nu-h,1}$, $j_{\nu,h,1}$, $j'_{\nu+m1}$, and $j_{\nu+m1}$.

By virtue of the graph of $\psi_\nu^h(z)$, it follows that $\psi_\nu'^h(j_{\nu,h,1}) < 0$ and hence using the criterion under section 4, we arrive at the inequality

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$$j'_{\nu,h,1} < j_{\nu,h,1} \tag{4c.1}$$

Also, using $(1\cdot1)$, it is obvious to see that

$$\psi_\nu{}^h(j_\nu) < 0, \tag{4c.2}$$

which, on using the criterion under section 4, implies that

$$j_{\nu,h,1} < j_\nu. \tag{4c.3}$$

Moreover, using $(3\cdot7)$, we find that

$$\psi'_\nu{}^h(j_\nu') = \left\{ \frac{\nu^2 - h}{j_\nu'^2} - 1 \right\} J_\nu(j_\nu'), \tag{4c.4}$$

which, by virtue of $(4b\cdot1)$, furnishes

$$\psi_\nu'{}^h(j_\nu') < 0, \tag{4c.5}$$

and hence, using the criterion under section 4, we obtain

$$j'_{\nu,h,1} < j\nu' \tag{4c.6}$$

Again, using the results[1]

$$j'_{\nu+m_1} < j'_{\nu+m_2} \text{ and } j_{\nu+m_1} < j_{\nu+m_2} \tag{4c.7}$$

for positive integral values of m_1 and m_2, such that $0 \leqslant m_1 < m_2$; $(4c.3)$ and $(4c.6)$ may be extended to the forms

$$j'_{\nu,h,1} < j'_{\nu+m_1} \tag{4c.8}$$

and

$$j_{\nu,h,1} < j_{\nu+m_1}. \tag{4c.9}$$

4. *Relation involving* $j_{\nu,h,1}$ *and* $j_{\nu+m,h,1}$

By virtue of $(1\cdot3)$, it follows that

$$j_{\nu-1} \leqslant j_{\nu,h,1} < j_\nu, \tag{4d.1}$$

for $h \geqslant \nu > 1$; which on extending, assumes the form

$$j_\nu \leqslant j_{\nu+1,h,1} < j_{\nu+1}, \tag{4d.2}$$

for $h \geqslant \nu + 1$.

Combining $(4c.1)$ and $(4c.2)$, we arrive at the inequality

$$j_{\nu,h,1} < j_{\nu+1,h,1}, \tag{4d.3}$$

provided that $h \geqslant \nu + 1$.

In general, it can be shown that

$$j_{\nu+m_1,h,1} < j_{\nu+m_2,h,1}, \tag{4d.4}$$

for positive integral values of m_1 and m_2, such that $0 \leqslant m_1 < m_2$; provided that $h \geqslant \nu + m_2$.

5. Idea of the existence of $j''_{v,h,1}$

Combining (3·7) and (4·3), we get

$$z^2 \, \psi''_v{}^h(z) = -(h-1) \, J_v'(z) - \{v^2 - h - (h+v)(v-1)^2 + z^2(h+v-1)\} \, \frac{J_v(z)}{z}$$

$$+ \{z^2 - (v^2+1-2h)\} \, J_{v+1}(z). \tag{5·1}$$

Replacing z by $\sqrt{v^2+1-2h}$, (5 1) reduces to the form

$$(v^2+1-2h) \, \psi''_v{}^h(\sqrt{v^2+1-2h}) = -(h-1) \, J'_v(\sqrt{v^2+1-2h})$$

$$- \frac{2 \, J_v(\sqrt{v^2+1-2h})}{\sqrt{v^2+1-2h}} \left[v^2 - \left\{ h^2 - \frac{h}{2} + \tfrac{1}{2} \right\} \right]. \tag{5·2}$$

Again, by virtue of (4b.1) and (4b.2), we find that

$$J_v' \, (\sqrt{v^2+1-2h}) > 0 \quad \text{and} \quad J_v(\sqrt{v^2+1-2h}) > 0, \tag{5·3}$$

so long as $h \geqslant v > 1$; and thus the right hand side of (5·2) is a negative quantity

for

$$v^2 > h^2 - \frac{h}{2} + \tfrac{1}{2}. \tag{5·4}$$

Hence, it may be concluded, by virtue of the criterion under 4, that

$$j''_{v,h,1} < \sqrt{v^2+1-2h,}{}^\dagger \tag{5·5}$$

subject to the restrictions $h \geqslant v > 2$ and (5·4).

5a. Special cases: In particular, for $h = v$, (5 5) reduces to the form

$$j''_{v-1} < v - 1, \tag{5a.1}$$

provided that $v > 2$; a result established previously[1].

5b. Relation between $j''_{v,h,1}$ and j_v''

It is obvious to see that $v^2 - v > v^2 + 1 - 2h$ for $h \geqslant v > 1$.

Hence, using the inequality[1]

$$j_v''^2 > v^2 - v. \tag{5b.1}$$

we arrive at the inequality

$$j''_{v,h,1} < j_v'', \tag{5b.2}$$

provided that $h \geqslant v > 1$ and the restriction given by (5·4).

6. Signs of $\psi_{v+m}^h \, (j'_{v,h,1})$ and $\psi_{v+m}^h \, (j_{v,h,1})$ for positive integral values of m.

By virtue of (1·1), (4b.1) and (4c.8), we arrive at the result

$$\psi_{v+m}^h \, (j'_{v,h,1}) > 0. \tag{6·1}$$

$\dagger \psi_v''^h(z)$ is negative for sufficiently small values of 'x' for $0 < v < 2$

Also, by virtue of (4c.9), it follows that

$$\psi_{\nu+m}^{h}\ (j_{\nu,h,1}) > 0, \tag{6.2}$$

provided that $h \geqslant \nu + m$.

It is, however, interesting to examine the range of values of h and ν, for which $\psi_{\nu+m}^{h}\ (j_{\nu,h,1}) > 0$, subject to the restriction (5.4).

Combining (1.3) and (3.2), we find

$$\psi_{\nu+m}^{h}(z) = \frac{(h-\nu)}{z}\ J_{\nu+m}(z) + \frac{m}{z}\ \psi_{\nu+m-1}^{h}(z) + \left\{ 1 - \frac{m(h+\nu+m-1)}{z^2} \right\} J_{\nu+m-1}(z) \tag{6.3}$$

Replacing z by $j_{\nu,h,1}$ (6.3) changes to the form

$$\psi_{\nu+m}^{h}\ (j_{\nu,h,1}) = \frac{(h-\nu)}{j_{\nu,h,1}}\ J_{\nu+m})(j_{\nu,h,1}) + \frac{m}{j_{\nu,h,1}}\ \psi_{\nu+m-1}^{h}\ (j_{\nu,h,1}) + \left\{ 1 - \frac{m(h+\nu+m-1)}{j^2_{\nu,h,1}} \right\} \times$$
$$J_{\nu+m-1}\ (j_{\nu,h,1}) \tag{6.4}$$

In particular, if $\psi_{\nu+m-1}^{h}\ (j_{\nu,h,1}) \geqslant 0$, then, by virtue of (4c.9), we obtain

$$\psi_{\nu+m}^{h}\ (j_{\nu,h,1}) > 0, \tag{6.5}$$

provided that $j^2_{\nu,h,1} > m(h+\nu+m-1)$ and $h \geqslant \nu$.

Now, using (5.4) and $h \geqslant \nu$, we find

$$\nu^2 + 1 - 2h - m(h+\nu+m-1) > h^2 - h(2m+\tfrac{5}{2}) - \{ m(m-1) - \tfrac{3}{2} \} \tag{6.6}$$

Obviously, the right hand side of (6.6) is a positive quantity, if 'h' is greater than or equal to the greatest root of the equation

$$x^2 - x(2m+\tfrac{5}{2}) - \{ m(m-1) - \tfrac{3}{2} \} = 0. \tag{6.7}$$

Or, in otherwords

$$h \geqslant (m+4) \left[1 + \left\{ 1 + \frac{4m^2 - 4m - 6}{(2m+\frac{5}{2})^2} \right\}^{\tfrac{1}{2}} \right] \simeq \frac{4m(5m+9)+19}{8m+10}. \tag{6.8}$$

Thus,

$$\nu^2 + 1 - 2h > m(h+\nu+m-1), \tag{6.9}$$

under the conditions (6.8), and therefore using (4a.2) we obtain

$$j^2_{\nu,h,1} > m(h+\nu+m-1). \tag{6.10}$$

Hence $\quad \psi_{\nu+m}^{h}\ (j_{\nu,h,1}) > 0$, under the conditions (6.8).

In particular, for $m = 1, 2, 3, 4$, we find

$$\psi_{\nu+1}^{h}\ (j_{\nu,h,1}) > 0 \quad \text{when} \quad h \geqslant 4.167 \text{ nearly}, \tag{6.11}$$
$$\psi_{\nu+2}^{h}\ (j_{\nu,h,1}) > 0 \quad \text{when} \quad h \geqslant 6.57 \text{ nearly}, \tag{6.12}$$
$$\psi_{\nu+3}^{h}\ (j_{\nu,h,1}) > 0 \quad \text{when} \quad h \geqslant 9.02 \text{ nearly}, \tag{6.13}$$

and $\quad \psi_{\nu+4}^{h}\ (j_{\nu,h,1}) > 0 \quad \text{when} \quad h \geqslant 11.5 \text{ nearly}. \tag{6.14}$

The results (6.11) – (6.14), have been checked, by means of tables of Bessel functions[5] as furnished by Section 13.

7. Inequalities satisfied by $j'_{\nu,h,1}$

Replacing 'z' by $j'_{\nu,h,1}$, (3·5) assumes the form

$$\left\{ \frac{h^2 - \nu^2 + j'^2_{\nu,h,1}}{h\, j'_{\nu,h,1}} \right\} J'_\nu(j'_{\nu\, h,1}) = \frac{j'_{\nu,h,1}}{h}\left\{ 1 - \frac{\nu^2 - h}{j'^2_{\nu,h,1}} \right\} \psi^h_\nu(j'_{\nu,h,1}) \tag{7·1}$$

Obviously, by virtue of (4c.1) and 4c.6), we find that

$$J'_\nu(j'_{\nu,h,1}) > 0 \text{ and } \psi_\nu^h(j'_{\nu,h,1}) > 0, \tag{7·2}$$

and hence it implies from (7·1) that

$$j'^2_{\nu,h,1} > \nu^2 - h, \tag{7·3}$$

provided that $\quad h \geqslant \nu > 0$.

In particular, for $h = \nu$, (7·3) reduces to the well known result[1]

$$j'^2_{\nu-1} > \nu^2 - \nu. \tag{7·4}$$

Moreover, replacing z by $j'_{\nu,h,1}$ (3·11) reduces to the form

$$\frac{h-\nu}{h\, j'_{\nu,h,1}}\left\{ h\, J_{\nu+2}(j'_{\nu,h,1}) + J'_\nu(j'_{\nu,h,1}) \right\} + J'_{\nu+1}(j'_{\nu,h,1}) + \frac{4(\nu+1)(h-\nu)}{j'^3_{\nu,h,1}} J_{\nu+1}(j_{\nu,h,1})$$

$$= \frac{-2\nu}{j'_{\nu,h,1}}\left\{ \frac{\nu^2-1}{j'^2_{\nu,h,1}} - \frac{h-\nu+2\nu h}{2\nu h} \right\} \psi^h_\nu(j'_{\nu,h,1}). \tag{7·5}$$

Obviously, by virtue of (4c.6), (4c.8) and (4c.9), it follows that

$$J_\nu'(j'_{\nu,h,1}) > 0, \; J'_{\nu+1}(j'_{\nu,h,1}) > 0, \; J'_{\nu+2}(j'_{\nu,h,1}) > 0 \text{ and } J_{\nu+1}(j'_{\nu,h,1}) > 0,$$

with the addition that $\quad \psi^h_\nu(j'_{\nu,h,1}) > 0$; and hence it implies from (7·5) that

$$j'^2_{\nu,h,1} > \frac{2\nu h(\nu^2-1)}{2\nu h + h - \nu}, \tag{7·6}$$

provided that $\quad h \geqslant \nu > 0$.

Also, replacing z by $j'_{\nu,h,1}$, (3·8) may be expressed in the form

$$\psi^h_{\nu+2}(j'_{\nu,h,1}) + \frac{2}{j'_{\nu,h,1}} J_{\nu+2}(j'_{\nu,h,1}) + \frac{2(h^2-\nu^2)}{h\, j'^2_{\nu,h,1}} J'_\nu(j'_{\nu,h,1})$$

$$= \left\{ 2\nu^2\frac{(h-1)}{h} - j'^2_{\nu,h,1} \right\} \frac{\psi^h_\nu(j'_{\nu,h,1})}{j'^2_{\nu,h,1}} \tag{7·7}$$

By virtue of (4c.6), (4c.7) and (6·1), the left hand side of (7·7) becomes positive for $h \geqslant \nu$, and $\psi^h_\nu(j'_{\nu,h,1}) > 0$; so that $j'_{\nu,h,1}$ satisfies the inequality

$$j'_{\nu,h,1} < 2\nu^2\frac{(h-1)}{h}, \tag{7·8}$$

provided that $h \geqslant \nu$ and $h > 1$.

Hence, combining (7·6) and (7·8), we arrive at the result

$$\left\{ \frac{2\nu h(\nu^2-1)}{2\nu h + h - \nu} \right\}^{\frac{1}{2}} < j'_{\nu,h,1} < \left\{ \frac{2\nu^2}{h}(h-1) \right\}^{\frac{1}{2}}, \tag{7·9}$$

[135]

provided that $h \geqslant \nu > 1$.

In particular, for $h = \nu$, (7·9) reduces to the well known form[1]

$$\sqrt{\nu^2 - 1} < j'_{\nu-1} < \sqrt{2\nu(\nu - 1)}, \tag{7·10}$$

which proves the validity of (7·9).

8. *Inequalities satisfied by* $j_{\nu,h,1}$.

By virtue of the results[1]

$$j_{\nu-1} > \sqrt{\nu^2-1}, \tag{8·1}$$

$$j_{\nu-1} > \sqrt{(\nu-1)(\nu+2)}, \quad (\nu > 5) \tag{8·2}$$

and (4d.1), we find

$$j_{\nu,h,1} > \sqrt{\nu^2-1}. \tag{8·3}$$

and

$$j_{\nu,h,1} > \sqrt{(\nu-1)(\nu+2)} \quad (\nu > 5) \tag{8·4}$$

Also, combining the results[1]

$$j_\nu < \sqrt{2\nu(\nu+1)} \tag{8·5}$$

and

$$j_\nu < \sqrt{\tfrac{4}{3}(\nu+1)(\nu+5)}, \tag{8·6}$$

with (4d.1), we get

$$j_{\nu,h,1} < \sqrt{2\nu(\nu+1)} \tag{8·7}$$

and

$$j_{\nu,h,1} < \sqrt{\tfrac{4}{3}\nu(\nu+1)(\nu+5)}, \tag{8·8}$$

Combining (8·3), (8·4), (8·7) and (8·8) we arrive at the inequalities,

$$\sqrt{\nu^2-1} < j_{\nu,h,1} < \sqrt{2\nu(\nu+1)}, \tag{8·9}$$

$$\sqrt{(\nu-1)(\nu+2)} < j_{\nu,h,1} < \sqrt{2\nu(\nu+1)}, \quad (\nu > 5) \tag{8·10}$$

and

$$\sqrt{(\nu-1)(\nu+2)} < j_{\nu,h,1} < \sqrt{\tfrac{4}{3}(\nu+1)(\nu+5)}, \tag{8·11}$$

provided that $h > \nu > 1$.

9. *Inequalities satisfied by* $j''_{\nu,h,1}$.

Replacing 'z' by $j''_{\nu,h,1}$, (3·16) reduces to the form

$$\psi^h_{\nu+1}(j''_{\nu,h,1}) + \frac{J_{\nu+1}(j''_{\nu,h,1})}{j''_{\nu,h,1}} + \frac{2(h^2-\nu^2)}{j''^2_{\nu,h,1}} J_\nu(j''_{\nu,h,1})$$

$$= \{j''^2_{\nu,h,1} - (\nu^2 + 2 - \nu - 2h)\} \frac{\psi^h_\nu(j''_{\nu,h,1})}{j''_{\nu,h,1}}. \tag{9·1}$$

Obviously, by virtue of (4c.1), (4c.9) combined with the result

$$j''_{\nu,h,1} < j'_{\nu,h,1}, \tag{9·2}$$

as follows obviously from the graphs of $\psi^h_\nu(z)$, we observe that the left hand side) of (9·1) is a positive quantity for $h \geqslant \nu$; and $\psi^h_\nu(j''_{\nu,h,1}) > 0$. Hence $j''_{\nu,h,1}$ satisfies the inequality

$$j''_{\nu,h,1} > (\nu^2 + 2 - \nu - 2h)^{\frac{1}{2}}. \tag{9·3}$$

[136]

Moreover, replacing z by $j''_{\nu,h,1}$ (3·17) assumes the form

$$\frac{h-\nu}{j''_{\nu,h,1}} J'_{\nu+1} (j''_{\nu,h,1}) + \left\{ 1 + \frac{(2\nu-1)(h^2-\nu^2)}{h j''^2_{\nu,h,1}} \right\} J_\nu'(j''_{\nu,h,1})$$

$$= (\nu-1) \left\{ \nu^2-2\nu + \frac{\nu(2\nu-1)(h-\nu)}{h(\nu-1)} - j''^2_{\nu,h,1} \right\} \frac{\psi^h_\nu(j''_{\nu,h,1})}{j''^2_{\nu,h,1}} , \qquad (9·4)$$

which implies, by virtue of (9·2) and (4c.9) that

$$j''_{\nu,h,1} < \left\{ \nu^2-2\nu + \frac{\nu(2\nu-1)(h-\nu)}{h(\nu-1)} \right\}^{\frac{1}{2}} , \qquad (9·5)$$

subject to the restrictions that $h \geqslant \nu > 1$.

Combining (9·3) and (9·5) we arrive at the inequality

$$(\nu^2+2-\nu-2h)^{\frac{1}{2}} < j''_{\nu,h,1} < \left\{ \nu^2-2\nu + \frac{\nu(2\nu-1)(h-\nu)}{h(\nu-1)} \right\}^{\frac{1}{2}} , \qquad (9·10)$$

provided that $h \geqslant \nu > 1$.

In particular, for $h = \nu$, (9·10) reduces to the well known form[1]

$$\sqrt{(\nu-1)(\nu-2)} < j''_{\nu-1} < \sqrt{\nu(\nu-2)}, \qquad (9·11)$$

proving the validity of the result (9·10).

10. *Numerical verification of the results under Section 4.*

Combining (3·3) and (3·7), we get

$$\psi_\nu'^h(z) = \left[\frac{(\nu-1)(h+\nu)}{z^2} - 1 \right] J_\nu(z) - \frac{h-1}{z} J_{\nu+1}(z). \qquad (10·1)$$

Moreover, combining (2·5) and (10·1), we obtain

$$z^2 \psi_\nu''^h (z) = [(h+\nu)(\nu-1)(\nu-2) - (h+\nu-1) z^2] \frac{J_\nu(z)}{z} + [z^2+3h-\nu^2-2] J_{\nu+1}(z) \qquad (10·2)$$

In accordance with the restrictions $h > \nu$, and $\nu^2+1 > 2h$, under section 4, we choose the pair of values $h = 4$ and $\nu = 3$ and evaluate $j'_{3,4,1}$ and $j''_{3,4,1}$ by means of the following Table.

<div align="center">TABLE I</div>

z	$\psi_3'^4(z)$	$\psi_3''^4(z)$	$\frac{d}{dz}[z \ \psi_3'^4(z)]$	Approximate value of $j'_{3,4,1}$	Approximate value of $j''_{3,4,1}$
1·5	. .	·0261	. .		
1·6	. .	−·0033	·3379	3·1	1·6
2 0	. .	. .	·0340		
3·0	·0397	. .	. .		
3·2	−·0238	. .	. .		

From the above consideration, it follows that $\psi_3^4(z)$ is a positive increasing function in the range $0 < z < 3·1$, which contains the range $0 < z < \sqrt{2}$, so that $\psi_3^4(z)$ is a positive increasing function in the range $0 < z < \sqrt{2}$. Also, $\psi''_3^4(z)$ remains positive for the values of z in the range $0 < z < \sqrt{2}$ which contains the range $0 < z < \sqrt{2}$. Thus $z \psi'_3^4(z)$ is a positive increasing function in the range $0 < z < \sqrt{2}$.

<div align="center">[137]</div>

Also, $z\,\psi'^4_3(z)$ and $\psi^4_3(z)$ are positive increasing functions in the range $\sqrt{l} \leqslant z \leqslant 2$. It is interesting to note that $\psi'^4_3(z)$ is a decreasing function in the range $1\cdot6 \leqslant z \leqslant 2$ and $\psi^4_3(z)$ remains a positive increasing function in $1\cdot6 \leqslant z \leqslant 2$.

11. *Numerical verification of* (4d.1) *and* (4d.4) *for* $m = 4$.

According to the condition $h \geqslant \nu + 4$, we choose the pairs $h = 6$; $\nu = 2$, and accordingly we compute the values of $\psi^h_\nu(z)$ in the form :

<div align="center">TABLE I</div>

z	$\psi_2^6(z)$	Range of $j_{2,6,1}$	$\psi_3^6(z)$	Range of $j_{3,6,1}$	$\psi_4^6(z)$	Range of $j_{4,6,1}$	$\psi_5^6(z)$	Range of $j_{5,6,1}$
4·4	·0266		. . . .		. . . .		. . . .	
4·5	−·0328	4·5	. . . .	5·6	. . . .	6·7	. . . .	7·7
5·5	. . . .	V	·0179	V	. . . .	V	. . . .	V
5·6	. . . .	$j_{2,6,1}$	−·0259	$j_{3,6,1}$	. . . .	$j_{4,6,1}$	. . . .	$j_{5,6,1}$
6·6	. . . .		. . . .		·0066		. . . .	
6·7	. . . .	V	. . . .	V	−·0270	V	. . . .	V
7·6	. . . .	4·4	. . . .	5·5	. . . .	6·6	·0248	7·6
7·7	. . . .		. . . .		. . . .		−·0029	

Obviously, from Table I, it follows that

$$j_{2,6,1} < j_{3,6,1} < j_{4,6,1} < j_{5,6,1}.$$

12. *Numerical verification of* (5 5) :

By virtue of the restrictions $\nu^2 + 1 > 2h$ and $\nu^2 > h^2 - \frac{h}{2} + \frac{1}{2}$, we choose the pair $h = 3\cdot1$ and $\nu = 3$, and accordingly compute the values of $z^2\,\psi''^{3\cdot1}_3(z)$, as given by

z	$z^2\psi_3''^{3\cdot1}(z)$	Range of $j''_{3,3\cdot1,1}$
1·5	·0360	$1\cdot5 < j_{3,3\cdot1,1} < 1\cdot6$
1·6	−·0259	

Considering (5·5), we observe that $0 < j''_{3,3\cdot1,1} < 1\cdot949$, which agrees with the range of $j''_{3,3\cdot1,1}$, as obtained in Table I.

13. *Numerical verification of* (6·11) − (6·14).

By virtue of the restrictions $h > 4\cdot16$, $6\cdot57$, $9\cdot02$, $11\cdot5$ and (5·4), we choose the respective pair of values $h = 4\cdot6$; $\nu = 4\cdot5$; $h = 6\cdot7$; $\nu = 6\cdot5$; $h = 9\cdot7$; $\nu = 9\cdot5$ and $h = 11\cdot7$; $\nu = 11\cdot5$; and accordingly compute the values of $\psi^{4\cdot6}_{5\cdot5}(7)$, $\psi^{6\cdot7}_{8\cdot5}(9\cdot45)$,

z	$\psi_{4\cdot5}^{4\cdot6}(z)$	Approximate value of $J_{4\cdot5,4\cdot6,1}$	$\psi_{5\cdot5}^{4\cdot6}(7)$	$\psi_{6\cdot5}^{6\cdot7}(z)$	Approximate value of $J_{6\cdot5,6\cdot7,1}$	$\psi_{8\cdot5}^{6\cdot7}(9\cdot45)$	$\psi_{9\cdot5}^{9\cdot7}(z)$	Approximate value of $J_{9\cdot5,9\cdot7,1}$	$\psi_{12\cdot5}^{9\cdot7}(12\cdot8)$	$\psi_{11\cdot5}^{11\cdot7}(z)$	Approximate value of $J_{11\cdot5,11\cdot7,1}$	$\psi_{15\cdot5}^{11\cdot7}(15\cdot05)$
7	·0006			. . . .			. . . .			. . . .		
7·1	−·0224			. . . .			. . . .			. . . .		
		7	·23328		9·45	·23671		12·8	·226		15·05	·1798
9·4	. . . .			·0026			. . . .			. . . .		
9·5	. . . .			− 024			. . . .			. . . .		
12·8	. . . .			. . . .			·0009			. . . ,		
15·05	. . . .			. . . .			. . . .			·0003		

Hence, we arrive at the conclusion that $\psi_{5}^{4\cdot6}{}_{5}(7) > 0$, $\psi_{8}^{6\cdot7}{}_{5}(9\cdot45) > 0$

$\psi_{12}{}_{5}^{9\cdot7}(12\cdot8) > 0$ and $\psi_{15}^{11\cdot7}{}_{5}$ (15·05) > 0, which proves the validity of (6·11)–(6·14).

14. *Numerical verification of* (7·3) *and* (7·9)

By virtue of the condition $h > \nu > 1$, we choose the pair $h = 3\cdot5$ and $\nu = 2$ and accordingly determine $j'_{2,3\cdot5,1}$, by the following method, as established previously[6].

14a. *Determination of* $j'_{2,3\cdot5,1}$:

Obviously (3·7) may be cast into the form

$$\psi'{}^{h}_{\nu}(z) = \frac{h-1}{z^2}\left[\left\{\frac{\nu^2-h}{h-1} - \frac{z^2}{h-1}\right\} J_\nu(z) + z\, J_\nu'(z)\right], \qquad (14a.1)$$

which is a transcendental function associated with even function

$$f(z) = \frac{\nu^2-h}{h-1} - \frac{z^2}{h-1}.$$

Following the symbols, as introduced previously[6], we find

$$A_0 = \frac{\nu^2-h}{h-1}, \ A_1 = -\frac{1}{h-1} \text{ and } A_i = 0 \text{ for integral values of } i \text{ greater than two,}$$

and thus we obtain

$$\sigma^{(1)}_{\nu,A_{i},1} = \frac{h+\nu+2}{4(h+\nu)(\nu\cdot1)}, \qquad (14a.2)$$

$$\sigma^{(2)}_{\nu,A_{i},1} = \frac{i}{(h+\nu)(\nu+1)(\nu-1)^2}\left\{1 + \frac{(\nu+1)(h+\nu-2)^2+4(h-1)^2}{2^4(h+\nu)(\nu+2)}\right\} \quad (14a.3)$$

and

$$\sigma^{(3)}_{\nu,A_{i},1} = \sigma^{(1)}_{\nu,A_{i},1}\left\{\sigma^{(2)}_{\nu,A_{i},1} - \frac{(\nu+3)(h+\nu+4)}{2^5(h+\nu)(\nu+1)(\nu+2)(\nu-1)}\right\}$$

[139]

$$+ \frac{(\nu-3)(\nu+2)+h(\nu+5)}{2^7(h+\nu)(\nu+3)(\nu+2)(\nu+1)(\nu-1)} + \frac{3}{2^6(h+\nu)(\nu+2)(\nu+1)(\nu-1)} \qquad (14a.4)$$

We evaluate $j'_{2,3\cdot5,1}$ by means of the following Table : $\qquad (h=3\cdot5, \nu=2)$

n	$\left[\sigma_{\nu,A_{i,1}}^{(n)}\right]^{-1/2n}$	$\left[\sigma_{\nu,A_{i,1}}^{(n)} \Big/ \sigma_{\nu,A_{i,1}}^{(n+1)}\right]^{k}$	$j'_{2,3\cdot5,1}$
1	1·713	2·188	
			1·95 nearly
2	1·935	1·961	
3	1·944		

Again, using the pair $h = 3\cdot5$, $\nu = 2$; (7·3) and (7·9) become

$$j'_{2,3\cdot5,1} > \cdot7071 \qquad (14a\cdot5)$$

and

$$1\cdot646 < j'_{2,3\cdot5,1} < 2\cdot39, \qquad (14a\cdot6)$$

which agree with the value $\quad j'_{2,3\cdot5,1} \approx 1\cdot95$.

15. *Numerical verification of* (8·10) *and* (8·11)

According to section 13, we choose the pair $h = 3\cdot5$, $\nu = 2$, and hence we obtain

$$\psi_2^{3\cdot5}(4\cdot1) = \cdot025 \quad \text{and} \quad \psi_2^{3\cdot5}(4\cdot2) = -\cdot0277, \qquad (15\cdot1)$$

which implies that $\quad 4\cdot1 < j_{2,3\cdot5,1} < 4\cdot2 \qquad (15\cdot2)$

Substituting the value $\nu = 2$, in (8·10) and (8·11), we get

$$1\cdot732 < j_{2,3\cdot5,1} < 5\cdot477 \qquad (15\cdot3)$$

and

$$2\cdot345 < j_{2,3\cdot5,1} < 5\cdot292, \qquad (15\cdot4)$$

which agree with (15·2).

16. *Numerical verification of* (9·10)

Choosing $h = 4$, $\nu = 3$, we obtain, by virtue of (10·2), that

$$(1\cdot6)^2 \ \psi_3''^4(1\cdot6) = -\cdot0085, \qquad (16\cdot1)$$

so that $j''_{3,4,1} \approx 1\cdot6$, which obviously lies in the range

$$0 < j_{3,4,1} < 2\cdot208, \qquad (16\cdot2)$$

as given by $\quad$ (9·10) for $h = 4$ and $\nu = 3$.

Acknowledgement

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Copper (II)-8-Hydroxyquinoline-7-Sulfonic Acid Chelate. A Spectrophotometric Study

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Abstract

Copper (II) forms in aqueous medium a colorless complex with 8-hydroxy-quinoline-7-sulfonic acid with maximum absorbance at 253 nm., stable in the pH range 1 to 3. The composition of the complex as established by the three different method spectrophotometrically is $Cu(R)_2$. (Where R stands for the ligand). The stability of the complex as calculated by the method of Banerji and Dey and the mole ratio method was found to be log $K-9\cdot8 \pm 0\cdot05$ at 25°C. The free energy of formation of the complex was found to be $-13\cdot5 \pm 0\cdot04$ K. Cal/mole. at 25°C. The effect of the ionic strength and temperature on the chelate has been investigated and the values found for $\triangle H$ and $\triangle S$ are $-5\cdot3$ K. Cal. and $63\cdot1$ e.u. respectively. The probable structure of the complex is also suggested.

Introduction

The ability of 8-hydroxyquinoline to form metal chelates is due to the periposition of the phenolic hydroxyl group to the hetero atom nitrogen, forming a grouping N-C-C-O which can give rise to the formation of five membered ring with a metal ion. Owing to the lack of selectivity, 8-hydroxyquinoline has been known to form insoluble chelates with at least thirty metal ions. On the other hand, the sulfonic acid derivatives of the reagents would be expected to form metal chelates, solubilised by the sulfonate group.

In the recent communication from this laboratory Banerji and Srivastava[1] reported the formation of 8-hydroxyquinoline 7-sulfonic acid derivatives and its analytical applications. Tiao Hsu Chang and Coworkers[2] have studied stability constant of the chelates of this reagent with Co(II), Zn(II), Ni(II), Pb(II) etc., but no significant work on its copper chelate has been reported. It has been considered worthwhile to study the interaction of Cu(II) and 8-hydroxyquinoline-7-sulfonic acid and to determine the stability constant spectrophotometrically.

Experimental

A standard solution of copper was prepared. Solution of 8-hydroxyquinoline-7-sulfonic acid was prepared by dissolving its sodium salt. Sodium perchlorate (E. Merk) was used to adjust the ionic strength. Absorbance measurements werh made on a HILGER Uvispec Spectrophotometer. (Model H 700-380) using one cm. effective light path. The cell compartment was fitted with a jacket

through which water can be circulated from a thermostat with variation of 0·01°C. Measurements of pH were made on a Beckman pH meter (Model H 2). All the measurements were made at 25°C and chelate formed was studied under the optimum conditions.

Results

Nature of the complex formed

To find out the nature of the complex, Vosburgh and Cooper method[3] was adopted. Mixtures containing various proportion of copper and the reagent (1:1, 1:2, 1:3, 1·4) were prepared and the absorbance at various wavelength were taken. The results have been graphically reported in fig. 1. It will be evident there from that the λ-max of the mixture is found to be in the spectral region of 253 nm. It is clear therefore that only one complex is formed under the conditions of study. The absorbance of the ligand shows λ-max at 238 nm.

Effect of pH on the chelate

Solutions containing copper and the reagent in the ratio 1:2 were prepared at different hydrogen ion concentration and the absorbance at the various wavelengths were taken. The complex was found to be stable in the pH range 1 to 3. It was found to retain its absorbance values over a wide range of temperature (25°C to 85°C).

Composition of the complex

The composition of the complex was studied by three different methods spectrophotometrically, Job's method of continuous variation, Mole ratio method and slope ratio method.

1. Job's method of continuous variation[4]

This method has been applied by taking equimolar and non-equimolar solutions of metal and the ligand and keeping the total volume at 25 ml. The absorbance of the metal ion, reagent and the mixture were measured at the two different wavelengths of 253 nm. 255 nm, after adjusting the pH to 3.

2. Mole ratio Method[5]

In the determination of composition by mole ratio method, a series of solutions were prepared from $5 \times 10^{-5}M$ and $2·5 \times 10^{-5}M$ of the reagent and equimolar solution of metal ion in such a way that the mole ratio of the reagent to metal was from 1:0·2 to 1:5. A break is found to occur when the ratio, reagent to metal is 1:2.

3. Slope-ratio method[6]

Here the concentration of the variable component was varied from $1 \times 10^{-5}M$ to $2 \times 10^{-4}M$ in the presence of an excess concentration ($1 \times 10^{-4}M$) of the constant component, adjusting the pH to 3. The slope of the two straight lines provide Iron : Reagent ratio, 1:2.

Evaluation of stability constant

For the purpose of determining the stability constant of the complex, the calculations have been made by the method used by Banerji and Day[7], and mole-ratio method[5]. The stability constant of the complex log K–9·8 ± 0·05 at 25°C, and the free energy of formation of the complex was found to be – 13·5 ± 0·04 K. Cals/mole at 25°C.
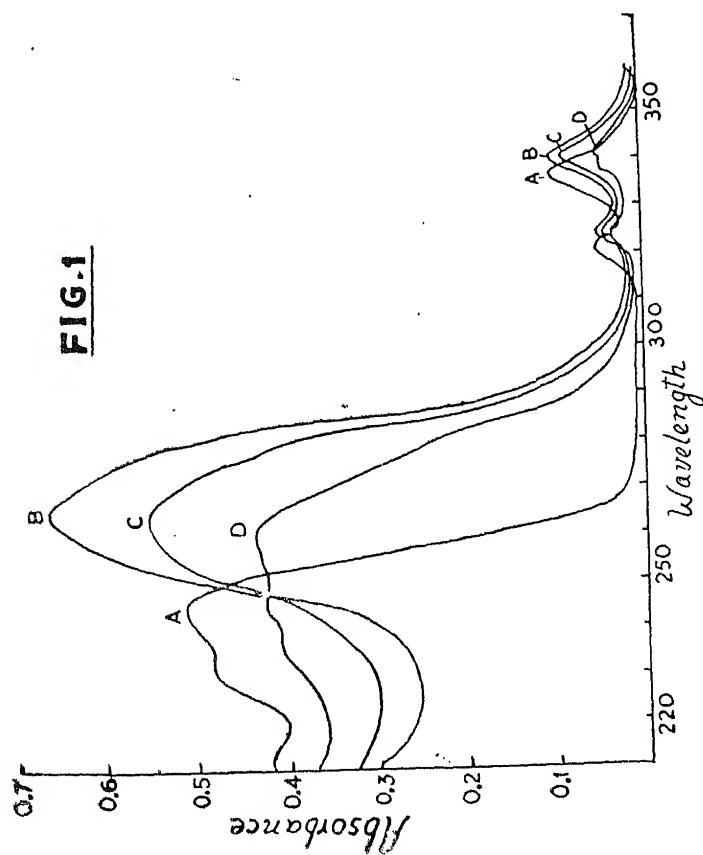
FIG.2

Fig. 2. Variation of log. K with temperature.

FIG.1

Fig. 1. Absorbance spectra of Copper –8–hydroxy-
quinoline–7–sulfonic acid chelate.

	Reagent 5×10^{-5}M	
Curve 1.	$C = 5 \times 10^{-5}$M,	$p = 1$.
Curve 2.	$C = 2 \cdot 5 \times 10^{-5}$M	$p = 2$.
Curve 3.	$C = 2 \cdot 5 \times 10^{-5}$M	$p = 2$.
Curve 4.	$C = 1 \cdot 6 \times 10^{-5}$M	$p = 3$.
Curve 5.	$C = 1 \cdot 25 \times 10^{-5}$M,	$p = 4$.
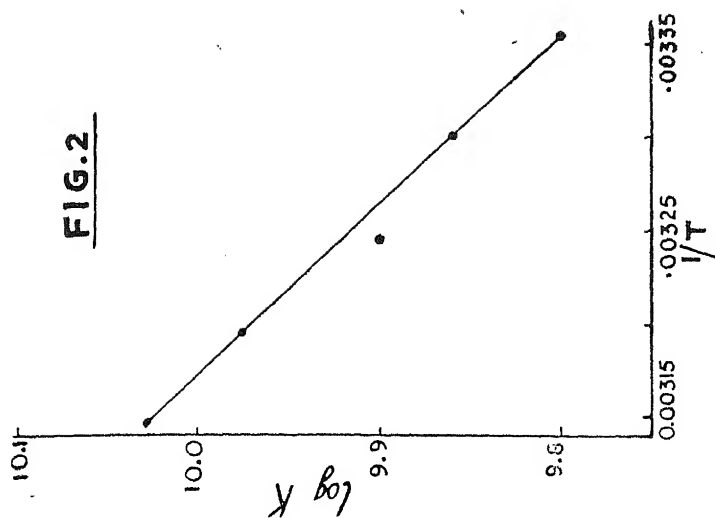
Entropy and enthalpy changes

In view of the difficulty in obtaining the thermodynamic stability constant, hence $\triangle F^{\circ}$, we have determined the entropy and enthalpy changes at a fixed ionic strenght. The stability constant of the complex at pH 3 and 0·02 ionic strength, at various temperature shown in Table 1.

TABLE I

Stability constants at different ionic strength. At temp. 25°C and pH 3.		Log. K at different temperatures, ionic strength, 0·02, pH 3.	
0·02	6·4 × 10⁹	25°C	9 8
0·05	5·5 × 10⁹	30°C	9·84
0·1	4·62 × 10⁹	35°C	9·89
0·2	3·7 × 10⁹	40°C	9·98
0·3	3·2 × 10⁹	45°C	10·03
0·5	3·18 × 10⁹		

From the slope of the curve obtained by plotting log K against 1/T, H of the reaction has been calculated. Assuming this to be constant over a range of experimental temperatures, $\triangle S$, of the reaction has been also calculated (Fig. 2).

Table 1 also records the stability constant of the complex at pH 3 at different ionic strengths at 25°C. The results show that stability constant decreases with an increase in the ionic strength, as expected, but beyond ionic strength 0·3 the value of K remains almost constant. This is probably due to the fact that the activity coefficient of an ion passes through a minimum as the ionic strength increases.

Discussion

Copper is a tetracoordinated bivalent ion, and 8-hydroxyquinoline-7-sulfonic acid, like its parent compound 8-quinolinol acts as a very strong chelating agent. During chelation the hydrogen of the hydroxyl group may be replaced by the number of metal ions (8, 9, 10) and the coordination link may then be obtained between the metal atom and the adjacent nitrogen atom. A five membered chelate ring can thus be formed in the structure of the complex molecule and a tentative structure of the 1:2 copper –8–hydroxy quinoline –7–sulfonic acid chelate can be represented as follows.

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Intrinsic Properties of Plane Diabatic Gas Flows

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Abstract

Considering two plane curves of congruences formed by the streamlines and their orthogonal trajectories in a plane of flow, the intrinsic properties of steady, diabatic gas flow are studied.

§1. Introduction : Several physical and chemical phenomena invalidate the assumption of adiabatic flow in many compressible flow problems. The inviscid non conducting steady gas flows with energy addition by heat sources are termed diabatic and corresponding to heating processes are thermodynamically reversible. The results from diabatic flow studies provide the basic insight into heat addition effects which is necessary. These diabatic flow problems were first formulated correctly in the independent work of Chapman[1] and Jouguet[4,5]. In the early part of this century Kiebel[6] gave a complete classification of viscous, compressible flow with energy addition into thirteen dynamically permissible categories. Hicks[3] formulating fundamental equations governing diabatic flows has considered intrinsic equations of plane flows. But the detailed investigations of flow parameters has yet to be considered. Consequently herein considering two plane curves of congruences, one related to streamline and other to its orthogonal trajectory. We have studied the intrinsic properties of plane diabatic gas flows.

§2. Fundamental Equation : The basic equations governing steady diabatic gas flow, in the absence of extraneous forces, in terms of Crocco's velocity vector field are given below in the usual notation[3].

$$(1) \qquad \text{div} \left\{ (1 - w^2)^{\frac{1}{\gamma-1}} \, \vec{w} \right\} = q \left(1 + \frac{\gamma+1}{\gamma-1} w^2 \right) (1 - w^2)^{\frac{2-\gamma}{\gamma-1}}$$

$$(2) \qquad \nabla \log p_t = \frac{2\gamma}{\gamma-1} \left(\frac{\vec{w} \wedge \text{curl } \vec{w} - q \vec{w}}{1 - w^2} \right)$$

$$(3) \qquad V_t \vec{w} \wedge \text{curl} \vec{w} = \dot{C}_p \nabla T_t - T \nabla S$$

$$(4) \qquad q = \vec{w}. \nabla \log T_t$$

$$(5) \qquad T_t = T (t - w^2)^{-1}$$

Where $\vec{w}$, q, γ, p_t, V_t, C_p, T_t, T and S are the reduced velocity vector, the heat content, the adiabatic exponent, the total pressure, the limiting velocity, the static temperature and the specific entropy respectively.

Geometric Relations : Considering $\vec{t}$ and $\vec{n}$ as tangent orthogonal unit vectors to the curves of congruences formed by the streamlines and their orthogonal trajectories in the plane of flow, and denoting d/ds, d/dn as directional derivatives along these vectors we have the following geometrical results[7] :

(6) $\quad \dfrac{d\vec{r}}{ds} = \vec{t} = \dfrac{\vec{w}}{w}$

(7) $\quad \dfrac{d\vec{t}}{ds} = \vec{n}\,K$ $\qquad$ (8) $\dfrac{d\vec{t}}{dn} = -\vec{n}\,k'$

(9) $\quad \dfrac{d\vec{n}}{ds} = -\vec{t}\,K$ $\qquad$ (10) $\dfrac{d\vec{n}}{dn} = \vec{t}\,K'$

where K and K' are the curvatures of the streamlines and their orthogonal trajectories.

Also considering a scalar-point function 'f' defined in the plane of flow, we obtain the following intrinsic condition to be satisfied by using the irrotationality character of 'grad f' as

(11) $\qquad K' \dfrac{df}{dn} + \dfrac{d}{dn}\left(\dfrac{df}{ds}\right) = K \dfrac{df}{ds} + \dfrac{d}{ds}\left(\dfrac{df}{dn}\right)$

This has to be satisfied by any flow quantity in a plane of flow. The same has been observed by Ericksen[2], for any function of class C''.

§3. **Intrinsic Properties of Fluid Flow** : In this section it is proposed to investigate the intrinsic properties of the diabatic compressible flow described above.

Making use of (5) in (1) we obtain

(12) $\qquad K' = \dfrac{1}{w(1 - w^2)}\left\{\dfrac{dw}{ds}(1 - \lambda w^2) + q(1 + \lambda w^2)\right\}$

where $\lambda = \dfrac{\gamma + 1}{\gamma - 1}$. The equation (12) gives the curvature distribution of orthogonal trajectories to the streamlines.

In the case of adiabatic gas flow the conservation of mass equation simplifies to

(13) $\qquad K' = \dfrac{1 - \lambda w^2}{w(1 - w^2)}\dfrac{dw}{ds}$

From this we observe that the streamlines can be either concentric circles or a family of parallel lines if and only if the magnitude of the velocity is uniform along an individual streamlines and the converse is also true.

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For Chapylingin's gas (*i.e.*, $\gamma = 0$) (12) simplifies to

(14)
$$K' = \frac{1}{w(1-w^2)} \left(q + \frac{dw}{ds} \right)$$

This gives the conservation of mass for diabatic Chapylingin's gas.

Also the reduced vorticity vector can be expressed as

(15)
$$\vec{\zeta} = \operatorname{curl} \vec{w} = \vec{b} \left(Kw - \frac{dw}{dn} \right)$$

where $\vec{b}$ is the unit vector perpendicular to the plane of flow.

Using (15) and (6) in (2) we obtain

(16)
$$\nabla \log pt = \frac{-2\gamma}{\gamma-1} \left\{ \vec{t} \frac{w\,q}{1-w^2} + \vec{n} \frac{w\left(Kw - \frac{dw}{dn} \right)}{1-w^2} \right\}$$

Now decomposing (16) we get

(17)
$$\frac{d}{ds} \log pt = \frac{-2\gamma\,q}{\gamma-1} \frac{w}{1-w^2}$$

(18)
$$\frac{d}{dn} \log pt = \frac{-2\gamma\,w}{(\gamma-1)(1-w^2)} \left(Kw - \frac{dw}{dn} \right)$$

These give us the variation of total pressure along the streamlines and their orthogonal trajectories. From (17) we observe that the total pressure is uniform along an individual streamline for adiabatic flow.

For Chapyligin's gas (17) and (18) simplify to

(19)
$$\frac{d}{ds} \log pt = \frac{q\,w}{w^2-1}$$

(20)
$$\frac{d}{dn} \log pt = \frac{w\,\zeta}{w^2-1}$$

From these we observe that the total pressure shall remain uniform throughout the region of flow for Chapyligin s gas if the fluid is adiabatic and flow potential. Also the total pressure shall decrease or increase in regions in which $w^2 \lessgtr 1$ accordingly.

Making use of (17) and (18) in (11) we obtain

(21)
$$\frac{w}{1-w^2}(K\,q - K'\,\zeta) + \frac{d}{ds}\left(\frac{w\,\zeta}{1-w^2} \right) = \frac{d}{dn}\left(\frac{q\,w}{1-w^2} \right)$$

This is the integrability condition to be satisfied in intrinsic form.

Expanding (21) and using (12) and (15) we obtain

(22)
$$\frac{d\zeta}{ds} + \frac{\zeta\,K'\,(1+w^2)}{1-\lambda\,w^2} = \frac{dq}{dn} + \frac{2\,w\,q}{1-w^2}\left\{ K\,w + \frac{\lambda\,\zeta\,(1+w^2)}{1-\lambda\,w^2} \right\}$$

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For Chapyligin's gas this simplifies to

(23)
$$\frac{d\zeta}{ds} + \zeta \, K' \, (1 + w^2) = \frac{dq}{dn} + \frac{2 \, q \, w^2}{1 - w^2}$$

Forming the scalar product of (3) by $\vec{t}, \vec{n}$ and using (4) and (5) we obtain

(24)
$$\frac{dS}{ds} = 2 \, q \, \frac{C_p \, (1 - w^2)}{w T^2}$$

(25)
$$\zeta = \frac{T}{w \, V \, t} \, \frac{dS}{dn} - \frac{C_p}{w \, V \, t} \left\{ \frac{\frac{dT}{dn}}{1 - w^2} + \frac{2 \, w \, T \, (K \, w - \zeta)}{(1 - w^2)^2} \right\}$$

From (24) we observe that the entropy shall remain uniform along a streamline in region, where the limiting velocity is equal to the velocity of the fluid, for diabatic gas flow. Also from (25) we obtain the vorticity of a fluid flow.

Also considering the inclination of the streamline with the fixed direction as 'θ' we have from (12)

(26)
$$\frac{d\theta}{dn} = \frac{1}{w \, (1 - w^2)} \left\{ \frac{dw}{ds} \, (1 - \lambda \, w^2) + q \, (1 + \lambda \, w^2) \right\}$$

From (2) we obtain

(27)
$$\frac{d\theta}{ds} = \frac{-(\gamma - 1) \, (1 - w^2)}{2 \, \gamma \, w^2} \frac{d}{dn} \, \log \left\{ p \, t \, (1 - w^2)^{\frac{\gamma}{\gamma - 1}} \right\}$$

These give variation of the slope of a streamline in terms of flow quantities.

§4. **Irrotational Field :** Herein we consider the intrinsic properties of reduced irrotational velocity vactor field and the flow quantities.

For irrotational flow from (15) we get

(28)
$$K = \frac{d\theta}{ds} = \frac{d}{dn} \, \log w$$

The equation (18) in this case simplifies to

(29)
$$\frac{d}{dn} \, \log p t = \frac{d}{dn} \, \log \left\{ p \, (1 - w^2)^{\frac{-\gamma}{\gamma - 1}} \right\}$$

Integrating this along an orthogonal trajectory to a streamline we obtain

(30)
$$p = C \, (1 - w^2)^{\frac{\gamma}{\gamma - 1}}$$

where 'C' is constant along an orthogonal trajectory to a streamline.

In this case (22) simplifies to

(31)
$$\frac{dq}{dn} + \frac{2 \, w^2 q \, k}{1 - w^2} = 0$$

Making use of (15) in (31) and integrating along the orthogonal trajectory of a streamline we get

$$(32) \qquad q = C_1 \, (1 - w^2)$$

where 'C_1' is the constant of integration.

From (30) and (32) we can find the pressure and velocity in terms of heating function.

Also from (3) we obtain

$$(33) \qquad \frac{dS}{dn} = \frac{d}{dn} \, (1 - w^2)^{-1} + \frac{C_p}{1 - w^2} \, \frac{d}{dn} \log T$$

This gives the variation of entropy along orthogonal trajectory to a streamline.

Summary

Considering two plane curves of congruences formed by the streamlines and their orthogonal trajectories, we arrived at the following results :

(1) Curvature distribution of orthogonal trajectories to the streamlines is expressed in terms of heat content and reduced velocity, from which adiabatic case can be deduced.

(2) For potential flows the total pressure shall remain uniform along the normals to the streamline.

(3) Integrability condition is derived in terms of the reduced velocity, the heat content and curvatures.

(4) The specific entropy shall remain uniform along a streamlines, in a region of flow at points the limiting velocity is equal to the velocity of the fluid.

(5) The fluid velocity and the pressure are expressed in terms of heat content for irrotational flows.

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Effect of Rock Phosphates on the Solubility of Various Sparingly Soluble Phosphates

By

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Abstract

It has been observed that secondary and tertiary phosphates of calcium, strontium, barium, magnesium, zinc, manganese, nickel. cobalt and ferric and aluminium phosphates are hydrolysed by water liberating small amounts of phosphoric acid in the solution and this hydrolysis is suppressed in presence of various Indian rock phosphates with all the sparingly soluble phosphates except that of ferric and aluminium phosphates, where a slight increase in the P_2O_5 solubility is noticed. The depressing capacity of rock phosphates on the solubility of various phosphates varies in the order of Rajasthan phosphate > Uttar Pradesh phosphate > Bihar phosphate.

Introduction

The effect of various salts namely NaCl, $NaNO_3$, KCl, KNO_3, NH_4Cl, NH_4NO_3 on the solubility of various sparingly soluble phosphates has been studied by several workers[2,4,9] during the last few years. The influence of iron and aluminium content of phosphate rock on the phosphorus solubility of superphosphate has been reported by Marshall. and Hill[8]. Verma[10] reported the effect of $CaCO_3$ on the solubility of calcium, barium, strontium, zinc and ferric phosphates. Islam and Rahman[7] observed that the availability of soil phosphorus to plants increases by the application of $(NH_4)_2SO_4$, $CaSO_4$ and $MgSO_4$ while $FeSO_4$, $Al_2(SO_4)_3$ and K_2SO_4 decreases the phosphorus availability. In the recent years Dhar, Pant and Kapoor[3] have studied the effect of calcium oxide and calcium carbonate on the P_2O_5 solubility of sparingly soluble phosphates, rock phosphates and basic slags.

For the use of rock phosphates in superphosphate manufacture phosphorus solubility in water is very important for its marketing. The seller must give a guarantee of minimum aluminium and iron impurities in the rock because the compounds of these elements in superphosphate lower the water solubility of superphosphate[6]. Recently some sizeable deposits of rock phosphates have been located in Rajasthan by Chatterji[1].

As no significant work has been reported on the effect of rock phosphates occurring in different parts of India, *e.g.* Rajasthan, Uttar Pradesh and Bihar on the solubility of various sparingly soluble phosphates, the present work has, therefore, been undertaken so as to assess the utility of these rock phosphates.

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Experimental

Rock phosphates from Rajasthan, Mussoorie (Uttar Pradesh) and Itagarh (Bihar) were first powdered and sieved through a 100 mesh sieve and then analysed in accordance with the procedure described by Washington[12].

All the phosphates used were prepared in the laboratory by precipitation method. Calculated amount of different phosphates for 0.5% P_2O_5 and 1 g of various sieved rock phosphates were taken in 250 ml stoppered pyrex flasks and 100 ml double distilled water was added to each of them. The contents of each flask were shaken for an hour in a mechanical shaker, allowed to stand for 24 hours in a thermostat maintained at 30°C and then filtered. Aliquot portions were utilized for determining P_2O_5, pH and electric conductivity[11].

A portable electronic pH–meter and magic eye conductivity bridge were utilized for pH and conductivity determinations respectively. The results are tabulated below :

TABLE 1
Composition of different rock phosphates

Ingrediants	RP %	MNP %	MPS %	MGP %	MVLP %	MRP %	I(B)RP %
Insoluble material	5.60	7.56	26.05	5.90	17.26	3.24	24.14
Al_2O_3	1.42	23.05	23.92	25.51	33.90	31.26	18.96
Fe_2O_3	0.28	0.25	0.24	0.25	0.32	0.96	12.17
CaO	44.98	32.15	17.00	16.45	15.50	11.38	9.81
MgO	0.08	0.09	0.06	0.05	0.05	0.04	0.35
P_2O_5	27.18	15.46	10.34	30.17	11.97	30.72	18.60

TABLE 2
Solubility of various phosphates and rock phosphates at 30°C

Phosphates	Solubility P_2O_5 in g/l.	Equilibrium pH	Specific conductivity
$Ca_3(PO_4)_2.5H_2O$	0.1475	5.80	2.65×10^{-4}
$CaHPO_4$	0.0562	7.60	1.38×10^{-4}
$Sr_3(PO_4)_2.2H_2O$	0.1174	7.90	2.99×10^{-4}
$SrHPO_4$	0.1072	7.20	1.86×10^{-4}
$Ba_3(PO_4)_2.2H_2O$	0.0545	7.30	0.98×10^{-4}
$BaHPO_4.2H_2O$	0.0430	7.50	0.65×10^{-4}
$Mg_3(PO_4)_2 5H_2O$	0.2345	7.45	3.80×10^{-4}
$MgHPO_4.3H_2O$	0.2815	7.55	4.25×10^{-4}
$Zn_3(PO_4)_2.4H_2O$	0.0650	7.30	1.48×10^{-4}
$ZnHPO_4.$	0.0596	7.45	0.99×10^{-4}
$Mn_3(PO_4)_2.7H_2O$	0.0113	7.15	0.49×10^{-4}

Phosphates	Solubility P_2O_5 in g/l	Equilibrium pH	Specific conductivity
$MnHPO_4.3H_2O$	0·0165	7·05	$0·60 \times 10^{-4}$
$Ni_3(PO_4)_2.7H_2O$	0·0162	7·45	$0·61 \times 10^{-4}$
$NiHPO_4$	0·0122	7·20	$0·45 \times 10^{-4}$
$Co_3(PO_4)_2.$	0·0189	7·65	$1·13 \times 10^{-4}$
$CoHPO_4$	0·0136	7·30	$0·75 \times 10^{-4}$
$AlPO_4.H_2O$	0·0615	7·35	$2·18 \times 10^{-4}$
$FePO_4.H_2O$	0·0402	6·90	$1·08 \times 10^{-4}$
RP	0·0052	7·25	$1·41 \times 10^{-4}$
MNP	0·0045	7·20	$1·37 \times 10^{-4}$
MPS	0·0012	7·10	$0·95 \times 10^{-4}$
MGP	0·0040	7·10	$1·05 \times 10^{-4}$
MVLP	0·0009	7·15	$0·69 \times 10^{-4}$
MRP	0·0033	7·10	$1·02 \times 10^{-4}$
I(B)RP	0 0025	7·15	$1·00 \times 10^{-4}$

TABLE 3

Effect of rock phosphates on the solubility of various phosphate at 30°C
(P_2O_5 in g/l)

Phosphates	RP	MNP	MPS	MGP	MVLP	MRP	I(B)RP
$Ca_3(PO_4)_2.5H_2O$	0·0371	0·0562	0·1420	0·1428	0·1430	0·1437	0·1445
$CaHPO_4$	0·0137	0·0182	0·0202	0·0252	0·0268	0·0284	0·0366
$Sr_3(PO_4)_2.2H_2O$	0·0112	0·0285	0·0383	0 0408	0·0587	0·0661	0·0742
$SrHPO_4$	0·0110	0·0145	0·0278	0·0321	0·0469	0·0513	0·0599
$Ba_3(PO_4)_2.2H_2O$	0·0413	0·0466	0·0480	0·0485	0·0490	0·0499	0·0508
$BaHPO_4.2H_2O$	0·0132	0·0140	0·0149	0·0152	0 0154	0·0156	0·0159
$Mg_3(PO_4)_2.5H_2O$	0·1879	0·1843	0·1886	0·1895	0·1904	0·2010	0·1702
$MgHPO_4.3H_2O$	0·2260	0·2242	0·2316	0·2428	0·2465	0·2594	0·2075
$Zn_3(PO_4)_2.4H_2O$	0·0097	0·0123	0·0127	0·0272	0·0282	0·0315	0·0348
$ZnHPO_4.$	0·0090	0·0110	0·0120	0·0250	0·0273	0·0298	0·0321
$Mn_3(PO_4)_2.7H_2O$	0·0093	0·0100	0 0105	0·0106	0·0107	0 0108	0·0110
$MnHPO_4.3H_2O$	0·0125	0·0138	0·0143	0·0145	0·0146	0·0148	0·0152
$Ni_2(PO_4)_2.7H_2O$	0·0052	0·0058	0·0062	3·0086	0·0092	0·0101	0·0108
$NiHPO_4$	0·0048	0·0052	0·0054	0·0070	0·0078	0·0086	0·0094
$Co_3(PO_4)_2$	0·0070	0·0080	0·0091	0·0098	0·0108	0·0115	0·0125
$CoHPO_4$	0·0060	0·0069	0·0084	0·0089	0·0095	0·0099	0·010 2
$AlPO_4.H_2O$	0·0845	0·0815	0·0801	0·0779	0·0600	0·0733	0·0708
$FePO_4.H_2O$	0·0589	0·0467	0·0435	0·0415	0·0405	0·0401	0·0365

TABLE 4

Effect of rock phosphates on the pH of various phosphates at 30°C

Phosphates	RP	MNP	MPS	MGP	MVLP	MRP	I(B)RP
$Ca_3(PO_4)_2 \cdot 5H_2O$	7·20	7·15	6·15	6·10	6 05	6·00	5·95
$CaHPO_4$	7·85	7·80	7·75	7·70	7·70	7·65	7·55
$Sr_3(PO_4)_2 \cdot 2H_2O$	8·90	8·75	8·35	8·25	8·20	8·15	8·05
$SrHPO_4$	7·70	7·65	7·55	7·50	7·45	7·40	7·35
$Ba_3(PO_4)_2 \cdot 2H_2O$	7·90	7 80	7·75	7·70	7·65	7·60	7·50
$BaHPO_4 \cdot 2H_2O$	7·95	7·85	7·80	7·75	7·70	7·65	7·60
$Mg_3(PO_4)_2 \cdot 5H_2O$	7·70	7·80	7·65	7·60	7·55	7·50	7·90
$MgHPO_4 \cdot 3H_2O$	7·90	8·00	7·80	7·75	7·70	7·60	8·05
$Zn_3(PO_4)_2 \cdot 4H_2O$	7·85	7·75	7·70	7·60	7·35	7 45	7 35
$ZnHPO_4$	8·00	7·80	7·75	7 75	7·60	7·55	7·50
$Mn_3(PO_4)_2 \cdot 7H_2O$	7·75	7 60	7·55	7·50	7·45	7·40	7·30
$MnHPO_4 \cdot 3H_2O$	7·60	7 50	7 45	7·40	7·35	7·30	7·25
$Ni_3(PO_4)_2 \cdot 7H_2O$	8·05	7·95	7·90	7·80	7·75	7·60	7·55
$NiHPO_4$	7·90	7·85	7·80	7·60	7·55	7·45	7·40
$Co_3(PO_4)$	8·20	8·05	7 90	7·85	7·80	7·75	7·70
$CoHPO_4$	7·95	7·80	7·70	7 65	7 60	7·55	7·50
$AlPO_4 \cdot H_2O$	8·15	8·00	7·90	7·85	7·30	7·45	7·40
$FePO_4 \cdot H_2O$	7·85	7 60	7·45	7·25	7·05	6·85	6·65

TABLE 5

Effect of rock phosphates on the specific conductivity of various phosphates at 30°C (Sp. cond. in mhos $\times 10^{-4}$)

Phosphates	RP	MNP	MPS	MGP	MVLP	MRP	I(B)RP
$Ca_3(PO_4)_2 \cdot 5H_2O$	2·06	2·18	2·27	2·30	2·31	2·33	2·34
$CaHPO_4$	0·60	0·65	0·69	0·77	0 89	0·97	1·05
$Sr_3(PO_4)_2 \cdot 2H_2O$	1·47	1·58	1·62	1·73	1·80	2·41	2·65
$SrHPO_4$	1·18	1·25	1·38	1·42	1·49	1·62	2·64
$Ba_3(PO_4)_2 \cdot 2H_2O$	0·70	0·79	0·87	0·88	0 90	0·92	0·94
$BaHPO_4 \cdot 2H_2O$	0·21	0·23	0·25	0·26	0·27	0·28	0·30
$Mg_3(PO_4)_2 \cdot 5H_2O$	3·06	2·97	3·05	3·07	3·09	3·27	2·74
$MgHPO_4 \cdot 3H_2O$	3·87	3·85	3·89	3·90	3·97	3·99	3·73
$Zn_3(PO_4)_2 \cdot 4H_2O$	1·02	1·05	1·06	1·16	1·28	1·34	1·42
$ZnHPO_4$	0·20	0·25	0·28	0·45	0·46	0·55	0·52
$Mn_3(PO_4)_2 \cdot 7H_2O$	0·40	0·42	0·44	0·45	0·46	0·47	0·48
$MnHPO_4 \cdot 3H_2O$	0·45	0·49	0·52	0·53	0·54	0·55	0·57
$Ni_3(PO_4)_2 \cdot 7H_2O$	0·38	0·41	0·43	0·49	0·50	0·53	0·55
$NiHPO_4$	0·18	0·19	0·20	0·25	0·27	0·32	0·36
$Co_3(PO_4)$	0·43	0·48	0·55	0·59	0·64	0·67	0·76
$CoHPO_4$	0·34	0·38	0·46	0·49	0·52	0·55	0·57
$AlPO_4 \cdot H_2O$	3·25	3·02	2·75	2·55	2·05	2·41	2·35
$FePO_4 \cdot H_2O$	1·34	1·22	1·18	1·16	1·10	1·05	1·03

RP = Rajasthan Phosphate. MNP = Mussoorie Nodular Phosphate. MPS = Mussoorie Phosphorite Shale. MGP = Mussoorie Granular Phosphorite, MVLP = Mussoorie Variegated Lenticular Phosphorite, MRP = Mussoorie Rock Phosphate, I(B)RP = Itagarh (Bihar) Rock Phosphate

[154]

Discussion

The analysis of various Indian rock phosphates (Table 1) shows that only a few rock phosphates are rich in the P_2O_5 content and the amount varies in the order of MRP > MGP > RP > I(B)RP > MNP > MVLP > MPS.

The analysis results further indicate that except RP all the other rock phosphates are richer in their alumina and ferric oxide content. This clearly indicates that except for RP other rock phosphates can not be directly utilized for the manufacture of super phosphate unless some benefication is done with these rock phosphates so as to reduce their ferric and aluminium oxide contents.

On studying the influence of various rock phosphates on the solubility of sparingly soluble phosphates (Table 3) it seems that they may be divided into two catagories. One group consisting of the phosphates of calcium, strontium, barium, magnesium, zinc, manganese, nickel and cobalt, in which the amount of phosphoric acid in the solution is depressed by the addition of rock phosphates. The influence of rock phosphates in decreasing the amount of phosphoric acid in solution varies in the order of RP > MNP > MPS > MGP > MVLP > MRP > I(B)RP with all the sparingly soluble phosphates except with magnesium phosphate. With magnesium phosphate the order is as I(B)RP > MNP > RP > MPS > MGP > MVLP > MRP.

It has been observed that when phosphates are dissolved in water they are hydrolysed liberating small amounts of phosphoric acid and the respective oxides in the solutions. When rock phosphates are added to the solutions of calcium and magnesium phosphates appreciable depression in the P_2O_5 solubility is noticed. It may be due to the fact that calcium and magnesium ions which are also present in the rock phosphates leads to the suppression of solubility due to common ion effect. Similar behaviour have been observed with strontium, barium, zinc, manganese, nickel and cobalt phosphates also, which may be attributed to the salting out effect[5].

It is interesting to note that the electric conductivity of the solutions (Table 5) invariably decreases and the pH- value (Table 4) increases in all these cases. These observations also support the decrease in the P_2O_5 solubility of the solutions.

The other group consisting of the aluminium and ferric phosphates, in which the free phosphoric acid in the solution is increased by the addition of all the rock phosphates except with MVLP in case of aluminium phosphate and with I(B)RP and MRP in case of ferric phosphate. MVLP and I(B)RP, MRP reduces the solubility of aluminium and iron phosphates respectively. This may be due to the fact that these rock phosphates are richer in aluminium and iron contents and due to common ion effect the hydrolysis in these particular instances is checked.

All the rock phosphates are slightly alkaline in nature and facilitate the hydrolysis of ferric and aluminium phosphates. As ferric and aluminium hydroxides are more insoluble than the phosphates, hydroxides are easily formed when OH ions are available in the system. The lime content of the rock phosphates dissolved appreciably in water and reacts with aluminium and ferric phosphates resulting in the formation of calcium phosphate, which leads to the greater P_2O_5 solubility

$$2AlPO_4 + 3Ca(OH)_2 \quad Ca_3(PO_4)_2 + 2Al(OH)_3$$

$$2FePO_4 + 3Ca(OH)_2 \quad Ca_3(PO_4)_2 + 2Fe(OH)_3$$

[155]
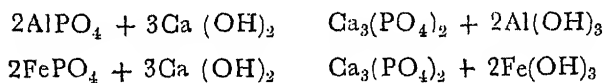
The pH and electric conductivity of the different solutions (Table 4 and 5) are in agreement with this view point.

It is clear from our experimental results that almost all the rock phosphates increase the amount of phosphoric acid passing into the solution with iron and aluminium phosphates. As the acidic soil invariably contains iron and aluminium compounds, the rock phosphates studied may be directly utilized as a fertilizer in the powdry form for better crop yields.

Acknowledgment

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K*–Curves of a Subspace of a Finsler Space

By

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1. Introduction

A curve relative to a vector field is called by Tsagas [3][1], as a special curve in a hypersurface of a Riemannian space, if the vector field is contained in the surface spanned by the geodesic curvature vectors of the curve. In this paper we have defined such a curve, relative to the congruences of curves, as K*-curve in a subspace of a Finsler space, and studied the properties of its curvature vector.

Let us consider a Finsler space F_m with coordinate system u^α [2] $(\alpha, \beta, \gamma \ldots = 1, 2, \ldots, m)$ as a subspace of a Finsler space $F_n(n > m)$ having coordinate system $x^i(i, j, k, \ldots = 1, 2 \ldots, n)$. The projection factor $B_\alpha{}^i$, regarded as a vector of F_n is given by

$$(1\cdot1) \qquad B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha},$$

and it is tangential to the coordinate curves of parameter u^α in F_m. The metric tensor $g_{\alpha\beta}(u, \dot{u})$ of F_m is related to the metric tensor $g_{ij}(x, \dot{x})$ of F_n by

$$(1\cdot2) \qquad g_{\alpha\beta}(u, \dot{u}) = g_{ij}(x, \dot{x}) B_\alpha^i B_\beta^j$$

As it is assumed that the rank of the matrix $\| B_\alpha{}^i \|$ is m, there are $(n-m)$ linearly independent vectors which are normal to F_m at a point P. There are two systems of normals, one depending on the directional argument $\dot{x}^i$ and other independent of it. In the present paper we have considered $(n-m)$ normals depending on the line–element which are given by the solutions of

$$(1\cdot3) \qquad g_{ij}(x, \dot{x}) \overset{*}{n}_{(\nu)}^j B_\alpha^i = \overset{*}{n}_{(\nu)i} B_\alpha^i = 0.$$

They satisfy the relations

$$(1\cdot4) \qquad g_{ij}(x, \dot{x}) \overset{*}{n}_{(\nu)}^i \overset{*}{n}_{(\mu)}^j = \delta_\mu^\nu \psi_{(\nu)}(x, \dot{x}),$$

(no summation with respect to ν)

and

$$(1\,5) \qquad g_{ij}(x, \overset{*}{n}_{(\nu)}) \overset{*}{n}_{(\nu)}^i \overset{*}{n}_{(\nu)}^j = 1,$$

---

(1) Numbers in the brackets refer to the references at the end of the paper.

(2) Early Greek indices $\alpha, \beta, \gamma, \ldots$, take the values $1, 2, \ldots, m$, while the latter Greek indices $\mu, \nu, \sigma, \ldots$, the values $m+1, \ldots, n$.

[157]

where $\psi_{(\nu)}$ is a scalar. These relations show that the secondary normals $\overset{*i}{n}_{(\nu)}$ form a system of mutually orthogonal unit vectors.

We consider the curve given by $x^i = x^i(s)$, where s is arc–length in F_m. For this curve under consideration the components of the first curvature vector with respect to F_n are given by

$$(1\cdot6) \qquad q^i = B_\alpha^i\, p^\alpha + \sum_\nu\, \Omega^*_{(\nu)\alpha\beta}\, \frac{du^\alpha}{ds}\, \frac{du^\beta}{ds}\, \overset{*i}{n}_{(\nu)}$$

where $p^\alpha = \dfrac{\delta u'^\alpha}{\delta s}$ are the components of its first curvature vector with respect to F_m which are given as

$$(1\cdot7) \qquad p^i = B_\alpha^i\, p^\alpha .$$

Here $\Omega^*_{(\nu)\alpha\beta}$ is secondary second fundamental tensor of the subspace.

2. Curvature of K^*–Curve. Let us consider a set of $(n-m)$ congruences of curves which are such that one curve of each congruence passes through each point of the subspace F_m. Suppose $\lambda^{*i}_{(\mu)}$ are the contravariant components of a unit vector in the direction of the curve of a congruence. This may be linearly expressed in terms of the tangent vector B^i_α and the system of normal $n^{*i}_{(\nu)}$ as[?],

$$(2\cdot1) \qquad \lambda^{*i}_{(\mu)} = t^{*\alpha}_{(\mu)}\, B_\alpha^i + \sum_\nu\, \Gamma^*_{(\mu\nu)}\, \overset{*i}{n}_{(\nu)},$$

where $t^{*\alpha}_{(\mu)}$ and $\Gamma^*_{(\mu\nu)}$ are the parmeters given by

$$(2\cdot2) \qquad t^{*\alpha}_{(\mu)} = B^\alpha_i\, \lambda^{*i}_{(\mu)}$$

and

$$(2\cdot3) \qquad \sum_\nu\, \Gamma^*_{(\mu\nu)}\psi_{(\nu)} = g_{ij}(x,\dot x)\, n^{*j}_{(\mu)}\, \lambda^{*i}_{(\mu)}.$$

They are related by the equations

$$(2\cdot4) \qquad g_{\alpha\beta}\, t^{*\alpha}_{(\mu)}\, n^*_{(\mu)i} = g_{\alpha\beta}\, B^\alpha_i \left(\sum_\nu\, \Gamma^*_{(\mu\nu)}\psi_{(\nu)} \right)$$

and

$$(2\cdot5) \qquad g_{\alpha\beta}\, t^{*\alpha}_{(\mu)}\, t^{*\beta}_{(\mu)} = \left(\sum_\nu\, \Gamma^*_{(\mu\nu)}\, \psi_{(\nu)} \right)^2$$

Definition $2\cdot1$. A K^* curve relative to the congruence $\lambda_{(\mu)}{}^*$ in a subspace is defined as a curve which has the property that at each point of the curve, the surface spanned by its first curvature vectors with respect to F_m and F_n contains the tangent to the curve of congruence $\lambda^*_{(\mu)}$ passing through the point.

As there are $(n-m)$ curves of $(n-m)$ congruences passing through a point, $(n-m)$ K^*–curves pass through a point with respect to the congruences of F_m. We can write $\lambda^{*i}_{(\mu)}$ in the following form

(2·6)
$$\lambda_{(\mu)}^{*i} = C_{(\mu)} \, p^i + D_{(\mu)} \, q^i,$$

where $C_{(\mu)}$ and $D_{(\mu)}$ are the parameters.

From (1·6), (1·7) and (2·1), the equation (2·6) becomes

(2·7)
$$t_{(\mu)}^{*a} B_a^i + \sum_\nu \Gamma_{(\mu\nu)}^* \, n_{(\nu)}^{*i} = (C_{(\mu)} + D_{(\mu)}) B_a^i \, p^a$$

$$+ \sum_\nu \Omega^*_{(\nu)a\beta} \, n_{(\nu)}^{*i} \, \frac{du^a}{ds} \frac{du^\beta}{ds} \, D_{(\mu)}.$$

Multiplying (2·7) by $g_{ij} B_\beta^j$ and using (1·2) and (1·3), we get

(2·8)
$$g_{a\beta} \, t_{(\mu)}^{*a} = g_{a\beta} \, p^a \, (C_{(\mu)} + D_{(\mu)}.$$

By virtue of (2·4), (2·8) gives

(2·9)
$$g_{a\beta} B_i^a \sum_\nu \Gamma^*_{(\mu\nu)} \, \psi_{(\nu)} = g_{a\beta} \, p^a \, n^*_{(\mu)i} \, (C_{(\mu)} + D_{(\mu)}).$$

Transvecting (2·8) by $g^{\beta\delta}$, we obtain

(2·10)
$$t^{*\delta}_{(\mu)} = (C_{(\mu)} + D_{(\mu)}) \, p^\delta.$$

From the equations (2·9) and (2·10), we have

(2·11)
$$p^\delta - g_{a\beta} \, p^a \, B^\beta_j \, n^{*j(\mu)} \, t^{*\delta}_{(\mu)} \left(\sum_\nu \Gamma^*_{(\mu\nu)} \, \psi_{(\nu)} \right)^{-1} = 0.$$

The solutions of the system of differential equations (2·11) determine the K^*-curves relative to the congruence $\lambda^{*i}_{(\mu)}$ in F_m. Let C be any curve of F_m defined by $u^a = u^a(s)$, where s is the arc-length. At any point of this curve we consider the vector

(2·12)
$$T^{*\delta}_{(\mu)} = p^\delta - g_{a\beta} \, p^a \, B^\beta_j \, n^{*j}_{(\mu)} \, t^{*\delta}_{(\mu)} \left(\sum_\nu \Gamma^*_{(\mu\nu)} \, \psi_{(\nu)} \right)^{-1},$$

which is defined as a K^*-curvature vector of the curve C relative to the congruence $\lambda^{*i}_{(\mu)}$ and the scalar $K^*_{(\mu)}$ given by

(2·13)
$$K^{*2}_{(\mu)} = g_{\gamma\delta} \, T^{*\gamma}_{(\mu)} \, T^{*\delta}_{(\mu)}$$

is called the K^*-curvature of the curve C.

From the equations (2·11), (2·12) and (2·13) we conclude that a K^* - curve of F_m in a F_n relative to a congruence may be defined as a curve along which the K^*-curvature vector relative to the congruence is a null vector.

We shall now find the K^* - curvature of the curve C relative to the congruence $\lambda^{*i}_{(\mu)}$. By virtue of (2·12) and (2·13), it is given by

[159]

$$(2\cdot14) \qquad K^{*2}_{(\mu)} = K^{*2}_g - 2\, g_{\alpha\beta}\, g_{\gamma\delta}\, p^\alpha\, B^\beta_j\, n^{*j}_{(\mu)}\, p^\gamma\, t^{*\delta}_{(\mu)}$$

$$+ \left(g_{\alpha\beta}\, p^\alpha\, B^\beta_j\, n^{*j}_{(\mu)} \right)^2 \left(g_{\gamma\delta}\, t^{*\gamma}_{(\mu)}\, t^{*\delta}_{(\mu)} \right) \left(\sum_\nu \Gamma^*_{(\mu\nu)}\, \psi_{(\nu)} \right)^{-2},$$

where K_g^* is the geodesic curvature of the curve C. The angle between the vector $t^*_{(\mu)}$ and p is given by

$$(2\cdot15) \qquad g_{\alpha\beta}\,(u,\dot u)\, t^{*\alpha}_{(\mu)}\, p^\beta = K^*_g\, (g_{\alpha\beta}\,(u,\dot u)\, t^{*\alpha}_{(\mu)}\, t^{*\beta}_{(\mu)})^{\frac{1}{2}} \cos \alpha_{(\mu)}$$

In view of the equations $(2\cdot4)$, $(2\cdot5)$ and (2.15), the equation $(2\cdot14)$ becomes

$$(2\cdot16) \qquad K_{(\mu)}^* = K_g^* \sin \alpha_{(\mu)}.$$

Thus we have

  Theorem $2\cdot1$ - The curve C is a K^* - curve provided

(a) it is a geodesic

or

(b) the vector $t_{(\mu)}^*$ is parallel to the vector p.

   We can express, the vector $t^{*\alpha}_{(\mu)}$, linearly in terms of the vectors $\dfrac{du^\alpha}{ds}$ and p^0 in the following manner

$$(2\cdot17) \qquad t^{*\alpha}_{(\mu)} = a_{(\mu)}\, \frac{du^\alpha}{ds} + b_{(\mu)}\, p^\alpha,$$

where $a_{(\mu)}$ and $b_{(\mu)}$ are the parameters to be determined.

We should note here that

$$(2\cdot18) \qquad g_{\alpha\beta}\, \frac{du^\alpha}{ds}\, \frac{du^\beta}{ds} = 1 \quad \text{and} \quad g_{\alpha\beta}\, \frac{du^\alpha}{ds}\, p^\beta = 0.$$

With the help of $(2\cdot5)$ and $(2\cdot18)$, $(2\cdot17)$ yields

$$(2\cdot19) \qquad \left(\sum_\nu \Gamma^*_{(\mu\nu)}\, \psi_{(\nu)} \right)^2 = a^2_{(\mu)} + b^2_{(\mu)}\, K^{*2}_g.$$

We also have

$$(2\cdot20) \qquad g_{\alpha\beta}\, t^{*\alpha}_{(\mu)}\, p^\beta = b_{(\mu)}\, K^{*2}_g,$$

which by virtue of $(2\cdot15)$ and $(2\cdot5)$ yields

$$(2\cdot21) \qquad \left(\sum_\nu \Gamma^*_{(\mu\nu)}\, \psi_{(\nu)} \right) \cos \alpha_{(\mu)}\, b =_{(\mu)} K^*_g.$$

Furthermore $(2\cdot17)$ gives

[160]

$$(2 \cdot 22) \qquad g_{\alpha\beta} \frac{du^{\beta}}{ds} t^{*\alpha}_{(\mu)} = a_{(\mu)}.$$

Since the angle between the vectors $t^{*\alpha}_{(\mu)}$ and $\frac{du^{\beta}}{ds}$ is given by

$$(2 \cdot 23) \qquad g_{\alpha\beta}(u, \dot{u}) t^{*\alpha}_{(\mu)} \frac{dv^{\beta}}{ds} = (g_{\alpha\beta}(u, \dot{u}) t^{*\alpha}_{(\mu)} t^{*\beta}_{(\mu)})^{\frac{1}{2}} \cos \beta_{(\mu)}.$$

With the help of $(2 \cdot 5)$, $(2 \cdot 18)$ and $(2 \cdot 23)$, $(2 \cdot 22)$ becomes

$$(2 \cdot 24) \qquad \sum_{\nu} \Gamma^{*}_{(\mu\nu)} \psi_{(\nu)} \cos \beta_{(\mu)} = a_{(\mu)}.$$

By virtue of the equations $(2 \cdot 21)$ and $(2 \cdot 24)$, the equation $(2 \cdot 19)$ yields

$$(2 \cdot 25) \qquad \operatorname{Sin} \alpha_{(\mu)} = \cos \beta_{(\mu)}.$$

Consequently the K^{*} — curvature of the curve C relative to the congruence $\lambda^{*i}_{(\mu)}$ is

$$(2 \cdot 26) \qquad K_{(\mu)}^{*} = K_{g}^{*} \cos \beta_{(\mu)}.$$

Thus we state

Theorem 2·2. The necessary condition for the curve C to be a K^{*}-curve is that the vector $t_{(\mu)}^{*\alpha}$ is orthogonal to it.

In the above results, if we put $m = n-1$, we get the K^{*}-curve of a hypersurface of a Finsler space with respect to the congruence of curves. In the case of hypersurface there is only one secondary normal n^{*i} and a congruence of curves, therefore there is only one K^{*}-curve passing through a point of the curve with respect to the congruence. For a hypersurface of a Finsler space, the equation of the K^{*}-curve, the K^{*}-curvature and its magnitude are given by

$$(2 \cdot 27) \qquad p^{\delta} - g_{\alpha\beta} p^{\alpha} B_{j}^{\beta} n^{*j} t^{*\delta} (\Gamma^{*}\psi)^{-1} = 0,$$

$$(2 \cdot 28) \qquad K^{*2} = K_{g}^{*2} - 2 g_{\alpha\beta} g_{\gamma\delta} p^{\alpha} B_{j}^{\beta} n^{*j} p^{\gamma} t^{*\delta} + (g_{\alpha\beta} p^{\alpha} B_{j}^{\beta} n^{*j})^{2}$$

and

$$g_{\gamma\delta} t^{*\gamma} t^{*\delta} (\Gamma^{*}\psi)^{-2}$$

$$(2 \cdot 29) \qquad K^{*} = K_{g}^{*} \sin \alpha,$$

respectively.

Identical results can be obtained for the normal vectors independent of directional argument.

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Heat Production in an Elliptic Cylinder

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1. Introduction

Cases in which heat is produced in the solid are important in technical applications [1, pp. 12–13]. Heat may be produced by the passage of an electric current. dielectric or induction heating, radioactive decay, absorption from radiation, mechanical generation in viscous or plastic flow, chemical reaction, hydration of cement and the ripening of apples. Nuclear reactors and space research also give rise to different problems of heat transfer.

The diffusion of heat in a cylinder when there are sources of heat within it which lead to an axially symmetrical temperature distribution has been already studied [Sneddon, 4]. No cylinder is perfectly circular, so to ascertain the influence of deviation from circularity, it is advisable to study the diffusion of heat in a cylinder of elliptical cross section.

In this paper we study diffusion of heat in an elliptic cylinder when there are sources of heat within it. To solve the problem we make use of a finite Matheiu transform, defined by Gupta [2]. It is known that an elliptic cylinder can be degenerated into a circular cylinder. We find that the result obtained by us when degenerated gives us the known result for the circular cylinder [4, p. 203.].

2. Differential equation

We treat the case where the elliptic cylinder is long enough in comparison with its major axis for its end effect to be neglected. If we take the interfocal line of the elliptic cylinder to be of length $2h$ and use elliptic coordinates (ξ, η) then the fundamental differential equation for the diffusion of heat in the cylinder, when there are sources within it, is of the form

$$\frac{\partial V}{\partial t} = \frac{2h^{-2}k}{(\cosh 2\xi - \cos 2\eta)}\left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2}\right) + \psi(\xi, \eta, t)$$

$$\begin{align} t &> 0 \\ 0 &< \xi < \xi_0 \\ 0 &\leqslant \eta \leqslant 2\pi \end{align} \qquad (1)$$

We further suppose that

$$\psi(\xi, \eta, t) = \frac{k}{K} f(\xi, \eta) g(t) \qquad (2)$$

where k is diffusivity, K is the conductivity of the material and the generation of heat is independent of temperature.

[162]

Now the equation (1) is to be solved with the boundary conditions.

$$V = 0 \quad \text{at} \quad \xi = \xi_0. \tag{3}$$

and
$$V = 0 \quad \text{when} \quad t = 0 \tag{4}$$

3.   A finite Mathieu transform

If any function $V(\xi, \eta)$ is continuous and single valued within ellipse and vanishes on the boundary surface then its finite Mathieu transform in the range $0 \leqslant \xi \leqslant \xi_0 ; \quad 0 \leqslant \eta \leqslant 2\pi$ is defined as [2, p. 779–780].

$$\overline{V}(q_{2n,m}) = \int_0^{\xi_0} \int_0^{2\pi} V(\xi, \eta)\, (\cosh 2\xi - \cos 2\eta) \times$$

$$Ce_{2n}(\xi, q_{2n,m})\, ce_{2n}(\eta, q_{2n,m})\, d\xi, d\eta. \tag{5}$$

where $\dot{q}_{2n,m}$ is the positive root of the equation

$$Ce_{2n}(\xi_0, q) = 0 \tag{6}$$

Its inversion is given by

$$V(\xi, \eta) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{\overline{V}(q_{2n,m})\, Ce_{2n}(\xi, q_{2n,m})\, ce_{2n}(\eta, q_{2n,m})}{\pi \int_0^{\xi_0} Ce^2_{2n}(\xi, q_{2n,m})\, [\cosh 2\xi - \theta_{2n,m}]\, d\xi} \tag{7}$$

where

$$\theta_{2n,m} = A_0^{(2n)}\, A_2^{(2n)} + \sum_{r=0}^{\infty} A_{2r}^{(2n)}\, A_{2r+2}^{(2n)} \quad [3, \text{p}, 177, \text{eqn. } 9] \tag{8}$$

and the sum is extended over the positive roots of the equation (6).

4.   Solution of the problem

Now applying the finite Mathieu transform (5) to (1) and using the result [2, eqn. 2·2]

$$\int_0^{\xi_0} \int_0^{2\pi} \left(\frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} \right) Ce_{2n}(\xi, q_{2n,m})\, ce_{2n}(\eta, q_{2n,m})\, d\xi.d\eta$$

$$= - 2\overline{V}\, q_{2n,m} \tag{9}$$

we get

$$\frac{d\overline{V}}{dt} + \frac{4\, q_{2n,m}}{h^2} \cdot k\, \overline{V} = \overline{f}(q_{2n,m})\, g(t) \cdot \frac{k}{K} \tag{10}$$

where
$$\overline{V} = 0 \quad \text{at} \quad t = 0 \tag{11}$$

and

$$\overline{f}(q_{2n,m}) = \int_0^{\xi_0} \int_0^{2\pi} f.\xi, \eta)\, (\cosh 2\xi - \cos 2\,\eta) \times$$

$$Ce_{2n}(\xi, q_{2n,m})\, ce_{2n}(\eta, q_{2n,m}).\, d\xi . d\eta. \tag{12}$$

Now using the condition (11) we obtain the solution of (10) as

$$\overline{v} = \frac{k}{K} \int_0^t e^{-k\lambda^2_{2n,m}\, (t-\tau)}\, \overline{f}(q_{2n,m})\, g(\tau)\, d\tau \tag{13}$$

where

$$\lambda^2_{2n,m} = \frac{4 \; q_{2n,m}}{h^2}.$$

(14)

Now from the inversion formula (7) we obtain

$$V(\xi, \eta, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{k}{K} \frac{Ce_{2n}(\xi, q_{2n,m}) \; ce_{2n}(\eta, q_{2n,m}) \times}{\pi \int_0^{\xi_0} Ce^2_{2n}(\xi, q_{2n,m}) \; [\cosh 2\xi - \theta_{2n,m}] \; d\xi}$$

$$\times \; \overline{f}(q_{2n,m}) \int_0^t e^{-k(t-\tau)} \; \lambda^2_{2n,m} \; g(\tau) \; d\tau.$$

(15)

where the sum is extended over the positive roots of the equation (6).

5. Transition to circular cylinder

If an elliptic cylinder degenerates into circular cylinder and V is independent of η then by [3, p. 330] we have

$$\lambda^2_{n,m} = \frac{\alpha^2_{0,m}}{a_2} \; , \; m = 1, 2, 3 \ldots \ldots \text{ etc.}$$

$\alpha_{0,m}$ being the roots of $J_0(\alpha_{0,m}) = 0$ Also

as $e \to 0$, as $\xi \to \infty$, $\sinh \xi \to \cosh \xi, h \cosh \xi \to r$ [as $h \to 0$],

$h \sinh \xi \; d\xi \to dr, \cosh 2\xi \; d\xi \to 2 \cosh \xi \; \sinh \xi \; d\xi, 2 \cosh h\xi \; \sinh \xi \; d\xi \to \dfrac{2r \; dr}{h^2}$,

using the results from [3, p. 330]

$$Ce_0(\xi, q_{0,m}) \to p_0' \; J_0 \left(\frac{\alpha_{0,m} r}{a} \right), \; [a \text{ is the radius of circular cylinder}].$$

$$ce_0(\eta, q_{0,m}) \to 2^{-\gamma_2}, \; A_0^{(0)} \to 2^{\gamma_2}, \; A_2^{(0)} \to 0, \; \theta_{2n} \to 0,$$

$$\lambda^2_{0,m} = \frac{\alpha^2_{0,m}}{a^2} = \frac{\alpha^2_m}{a^2}$$

Then the equation (15) degenerates into

$$V(r, t) = \frac{2k}{K} \sum_{m=1}^{\infty} \frac{\int_0^t g(\tau) \; e^{-k\alpha^2 m/a^2 (t-\tau)} \; d\tau.}{a^2 \; [J_1(\alpha_m)]^2} \; \; J_0(r \; \alpha_m/a) \times \overline{f}$$

(16)

where

$$\overline{f} = \int_0^a f(r) \; J_0 \left(\frac{\alpha_m r}{a} \right) r \; dr$$

(17)

Now taking $\alpha_m = p_m a$ in (16) we obtain a result which is same as given in [4, p. 203 eqn. (168)].

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A New 4-Phenylcoumarin 'Sisafolin' from *Dalbergia latifolia*

By

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Abstract

A new substituted 4-phenylcoumarin, sisafolin, has been isolated from the seeds of *Dalbergia latifolia*[1,2]. It has been assigned a tentative structure as 2', 7-dimethoxy-4', 5-dihydroxy-6-aldehydo-4-phenylcoumarin on the basis of its chemical and physical studies.
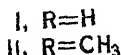
Sisafolin, the chief constituent of the seed of *Dalbergia latifolio*, has the molecular composition $C_{18}H_{14}O_7$, and molecular weight = 342 (Mass spectroscopy). It contains two methoxyl groups (ν_{max}^{KBr} 2950, 2830, 1460 and 1450 cm^{-1}). With dimethyl sulphate it forms a di-O-methyl ether, and with acetic anhydride and pyridine a diacetate showing the presence of two hydroxyl groups (ν_{max}^{KBr} 3425 cm^{-1}). These hydroxyl groups are phenolic, since there occurs a shift of 50 m μ in the U V spectrum of sisafolin on the addition of sodium ethoxide. It does not reduce Fehling's solution, but reduces Schiff's reagent and ammoniacal silver nitrate, showing thereby the presence of an aromatic aldehyde group (ν_{max}^{KBr} 1415, 1350, 1275 and 1160 cm^{-1}) which is further confirmed by the formation of a DNPH, m.p. 158°. Thus, the molecular formula of sisafolin can be written as $C_5H_5O_2 (OH)_2 (OCH_3)_2 (CHO)$.

Sisafolin dissolves in 20% aqueous caustic alkali and is reprecipitated on acidification, a test characteristic of lactones,

$$\left(\nu_{max}^{KB} \; 1200 \text{ cm}^{-1}, \right.$$
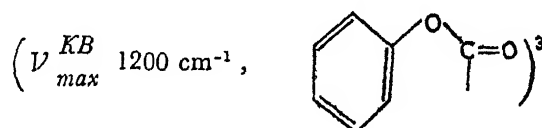
It gives a faint yellow colour with magnesium and acid (Shinoda test)[4], indicating it to be a 4-phenylcoumarin(ν_{max}^{KBr} 1740 cm^{-1}, δ-lactones and 4-phenylcoumarin)[5]. The faint colour in the above test is due probably to the presence of an aldehyde group in the ring[6]. Also, sisafolin on acetylation with acetyl chloride and sodium hydroxide gives an acid, m.p. 236-39°, which possesses three acetyl groups. The third acetyl group must have arisen by the acetylation of the -OH group formed by the opening of the lactone ring C. The mass spectrum of sisafolin shows peaks at m/e 342 (M), 311 (M-31), 176, 169, 133, 123, 10 , 89.

Based on the physical properties and the chemical reactions a probable structural assignment for sisafolin can be made as I, and for its di-O-methyl ether as II.

I, R=H
II, R=CH₃

Oxidation of sisafolin-O-dimethyl ether with neutral permanganate yields oxalic acid, β-resorcylic acid dimethyl ether and a substituted benzophenone. The DNPH, $C_{24}H_{22}O_{11}N_4$, m. p. 116° of the substituted benzophenone was found to contain four methoxyl and one carboxyl group. Peaks in the IR spectrum of the DNPH at 2850, 1398, 1320, 1285 and 1215 cm⁻¹ confirm the presence of carboxyl group in it. With ferric chloride its alcoholic solution gave violet colour indicating the presence of phenolic -OH group (ν_{max}^{KBr} 3400 cm⁻¹). Thus, the substituted benzophenone contains four methoxyl, one carboxyl and one hydroxyl group. Since the substituted benzophenone on further oxidation gave β-resorcylic acid dimethyl ether, it can be inferred that one ring of the substituted benzophenone has substitutions only at 2 and 4 positions by two methoxyl groups. The other benzene ring, therefore, must be substituted by the remaining two methoxyl, one carboxyl and one -OH groups. Peak at 910 cm⁻¹ in the IR spectrum of the DNPH of the substituted benzophenone indicates the other benzene ring to be pentasubstituted.

Thus the tentative structures of the substituted benzophenone and its DNPH may be written as III and IV respectively.

III

IV

The oxidation products, oxalic acid and β–resorcylic acid dimethyl ether, indicate that the $-\overset{\overset{\text{O}}{\|}}{C}-H$ group must be in the ring A of sisafolin. The position (C)8 for the aldehyde group is ruled out by the fact that with 20% aqueous caustic alkali sisafolin gives an acid from which the lactone is reprecipitated on acidification, which would not have been possible had there been a $-\overset{\overset{\text{O}}{\|}}{C}-H$ group at (C)8, due to hydrogen bonding between oxygen of the $\overset{H}{\diagdown}C=O$ group and the hydrogen of the -OH group (formed on the opening of the ring C with NaOH). Thus, the aldehyde group is most likely to be at position (C)6. The tentative positions for the two methoxyls are indicated to be (C)2' and (C)7 by the nmr spectrum (table 1) of sisafolin. This is also in agreement with Birch acetate rule for natural products.

Assuming I as the structure for sisafolin, the oxidation products of sisafolin--O-dimethyl ether can be explained satisfactorily :
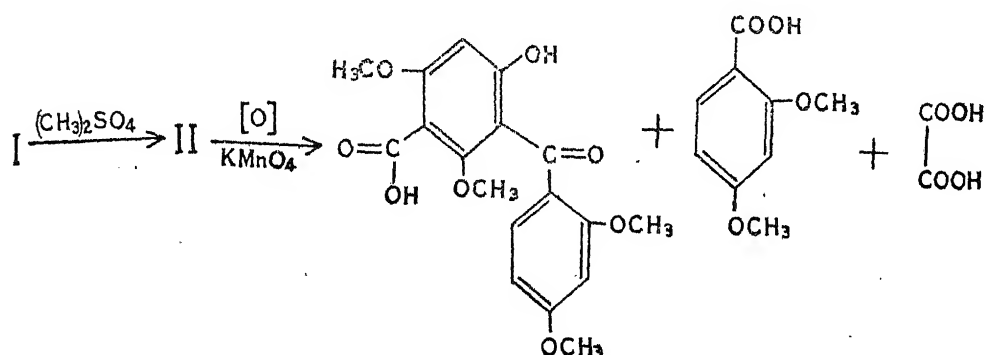
The nmr data (Table I) also supports the proposed structure.

TABLE I

NMR* spectral data of sisafolin

δ	Number of protons	Assignment
3·7	3	7 – OCH$_3$
3·95	3	2' – OCH$_3$
6	2	4', 5 – OH
6·72, 6·83	2	5', 6' - H
6·93	2	3, 8 – H
7·47	1	3' – H
8	1	6 - CHO

*The spectrum was recorded in Dimethyl sulphoxide with T. M. S. as an internal standard.

Experimental

Extraction of sisafolin

The seed of *D. latifolia* (4 Kg) was reduced to fine powder and then defatted with petroleum ether (40-60). The defatted material was then extracted with benzene in a soxhlet extractor. After concentration, the benzene extract (200 ml) was kept in a refrigerator when a white deposit was obtained. The deposit when chromatographed over a column of silica gel with benzene : chloroform (2:7) gave sisafolin, crystallising from methanol in faint yellow shining needles, m.p. 259-60°

(Found C, 63·12 ; H, 4·23%; $C_{18}H_{14}O_7$ requires C, 63·15 ; H, 4·11% ($\lambda_{max}^{ethanol}$ 308).

Sisafolin-O-dimethyl ether

Sisafolin (500 mg) was refluxed for 12 hr with dimethyl sulphate (2 ml), anhydrous K_2CO_3 (2·5 g) and acetone (10 ml). After filtration and removal of the acetone, the excess of dimethyl sulphate was decomposed by water. The solid product was extracted with ether, and the extract washed with 25 ml of aq. NaOH (1%). and finally with ether. The ether extract on evaporation gave a residue crystallising from methanol as granules, m.p. 230° (Found C, 64·28 ; H, 5·07%. Calculated for $C_{18}H_{12}O_5$ $(OCH_3)_2$ C, 64·86 ; H, 4·87%).

Acetylation of sisafolin

(a) With acetic anhydride and pyridine

Sisafolin (100 mg) was refluxed for 8 hr with acetic anhydride (10 ml) and pyridine (5 ml). The product was poured into ice cold water, when the acetyl derivative settled down. It was separated by filtration and crystallised from methanol. A granular colourless solid was obtained, m.p. 243-44° (Found C, 62·2; H, 4·32%. Calculated for $C_{22}H_{18}O_9$, C, 61·97 ; H, 4·22%).

(b) With acetyl chloride and sodium hydroxide

About 200 mg of the compound was dissolved in 15 ml of 10% NaOH and acetyl chloride (10 ml) was added to it. The mixture was kept at room temperature for 6 hr. The excess of acetyl chloride was then distilled off, the residue dissolved in water and the solution acidified with dil HCl. The precipitated acetyl derivative was extracted with $CHCl_3$, the solvent distilled off and the residue dissolved in methanol. On kseping the solution, crystals of acetyl derivatives parated, m.p. 239-40° (Found C, 58·76 ; H, 4·77%. Calculated for $C_{24}H_{22}O_{11}$, C, 59·26 ; H, 4·52%).

2 : 4-Dinitrophenylhydrazone of sisafolin

A solution of sisafolin (100 mg) in ethanol was added to a dilute acidic solution of 2 : 4-dinitrophenylhydrazine. The 2 : 4-dinitrophenylhydrazone crystallises out within 5 mts., m.p. 158° (Found C, 56·10 ; N, 10·82; H, 0·46%. Calculated for $C_{24}H_{18}O_{10}N_4$, C, 56·17 ; N, 10·73 ; H, 3·45%).

Oxidation of sisafolin-O-dimethyl ether

A solution of O-dimethyl sisafolin (500 mg) in acetone (100 ml) was treated with $KMnO_4$ (5 g) and left for 24 hr at room temperature. The acetone was distilled off and the residue suspended in water (100 ml) and saturated with sulphur dioxide. The mixture was extracted with ether and separated from

the mother liquor (A). The ether extract was washed with water, and extracted first with 5% aq. $NaHCO_3$ (B), then with 10% aq. NaOH (C), and finally with water.

The sodium bi-carbonate extract (B) on acidification, ether extraction and subsequent removal of the solvent gave an acid, $C_9H_{10}O_4$, m.p. 108° (34·06% –OCH_3) which was identified as β –resorcylic acid dimethyl ether. The sodium hydroxide extract (C) was worked up similarly and gave a small amount of an oily mass. It was found to be a substituted benzophenone and its DNPH crystallised from acetic acid as deep red elongated plates, m.p. 116°.

The mother liquor (A) was distilled under reduced pressure and the residue dried and extracted with hot ethyl acetate (3 × 25 ml). After removal of the solvent, a residue of oxalic acid was left which was crystallised from ether–light petroleum mixture as prisms (40 mg), m.p. and mixed m.p. 100–101°.

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Micro·detection of alkaloids in toxicological analysis

By

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Abstract

A paper chromatographic technique on salt buffered papers for separating alkaloids has been described. The fluorescent and absorbent spots observed under ultraviolet light, the colour of the spots with different spraying reagents, and the Rf. values of alkaloids are recorded. The pure samples of alkaloids separated and eluted from the chromatogram have been subjected for micro-crystal tests.

The technique for identifying alkaloids in the range of $1\mu g$ to $0.01\mu g$ by micro crystal tests, has been described. Details of the reagent used has also been recorded for identifying group of alkaloids.

The task of investigating deaths after the administration of plant alkaloids has long been a tedious one. For forensic purposes the identification of the poison is most desirable, and perhaps should be the prime aim of analysis. To confirm the diagnosis in the cases of suicides and murder by poisoning, the contents of the stomach, blood and urine samples in addition to parts of liver, spleen, kidney and piece of small intestine are examined for the presence of different poisons. Amorg these poisons, alkaloids are of special toxicological interest.

Analytical procedures for the detection of alkaloids are almost always based upon an organic-solvent extraction under specific pH conditions. Organic, non-volatile poisons are traditionally removed from autopsy tissues by a combination of extraction methods with final evaporation of a solvent to obtain the solid substance. Along with the poison is always extracted a certain amount of lipoidal material and a number of biological materials which will vary depending on the amount of putrefaction and other factors. As a rule extraction alone does not produce a well purified substance of the material in question.

In the past, alkaloids[1,2,3] and many miscellaneous drugs have been separated from each other and to a considerable extent identified on the basis of their chromatographic movement on a filter paper. Genest and Farmilo[4] surveyed the paper chromatographic methods for the identification of narcotics by the use of acid, alkaline and neutral solvents. Some workers use formamide, another polar substance, as an impregnant in combination with less polar mobile phases such as benzene, chloroform etc.[5,6,7]. Buffered salt papers were used by the author for the separation of alkaloids of Dhatura[8], papaver somniferum[9] and cocaine[10]. To a lesser extent, the alkaloids have been separated and effectively isolated by electrophoresis on paper[11-14]. A large number of alkaloids having practical value in toxicologic and forensic case work have been separated effectively by the author[15] using two specific buffer solutions. In both these procedures the isolated material ends in a spot on a strip of filter paper, where it is accessible for further identification (after elution for various colour and crystal test). In these techniques the

amount of the material being separated is necessarily very small, even when much larger amounts of the issues are available.

The procedure described in this paper takes advantage of the separation of alkaloids by paper chromatographic technique using buffered papers to achieve reproducibility of a very high degree, using a single developing system. Visualization of the alkaloids is achieved by employing compatible chromogenic sprays. The purified alkaloids eluted from the chromatogram are subjected to micro-chemical crystal tests for further specific identification.

Material and Methods

Reagents

The solvents and chemicals of B. D. H. reagent grade are used without further purification.

Whatman No. 1 chromatographic paper.

Developing solvent : Iso–butanol, 100 ml. ; glacial acetic acid, 10 ml; water 25 ml.

Buffered papers : I. pH 5·7, – ammonium sulphate solution, 2%, II. pH 6·6, – ammonium chloride solution, 2%.

Spray Reagents :

I. Modified Dragendorff reagent solution (A), 0·86 gm. of bismuth sub-nitrate dissolved in a mixture of 10 ml of glacial acetic acid and 40·0 ml. of water.

Solution (B) is prepared by dissolving 8 gm. of potassium iodide in 20 ml of water.

To prepare the spray solution, 5 ml. each of solutions (A) and (B) are fixed with 20 ml. of glacial acetic acid and 100 ml of water.

II. Iodoplatinate solution : 1 gm. of platinic chloride in 10 ml. of water is added to 60 gm. of potassium iodide in 200 ml. with water. It should be stored in the refrigerator, and fresh solution should be prepared every 2 weeks.

Preparation of buffered chromatographic papers

Whatman No. 1 chromatographic papers of required size were treated with solutions of ammonium sulphate (pH 5·7) and ammonium chloride (pH 6·6) and soaked dried between blotting papers to remove excess of solution. The papers were then allowed to dry for one hour by keeping them in a hot air oven at 60°C. The dried papers are stored at room temperature prior to use.

Alkaloid standards

Pure E. merck quality of alkaloids were dissolved in ethanol to give solutions containing 2 mg/ml.

Experimental—material :

29 alkaloids of forensic interest were selected for the study. On each Whatman buffered paper 2 to 4 μl of each sample, containing 8 to 10μg. of alkaloid, were spotted 2·5 cm from the one end of the paper by means of a microcapillary pipette. The spots were dried with a hot air blower.

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TABLE I
Paper chromatographic data for 29 alkaloids

| Alkaloid | Rf. value on papar buffered with | | Colour of spot in U.V. light (254 mμ) | Colour after spraying with | |
	Ammonium sulphate Rf × 100	Ammonium chloride Rf × 100		Modified Dragendorff reagent	Iodoplatinate reagent
Morphine	15·6	31·7	Blue	Orange	Blue
Codeine	25·0	43·0	Blue	Orange	Violet
Thebaine	52·0	73·7	Pink	Orange	Pink
Heroin	14·3	27·5	Bluish	Orange	Deep blue
Apomorphine	35·6	62·5	Black	Blackish	Pink
Narceine	60·0	68·7		Orange	Yellow
Yohimbine	62·5	78·1	Green	Brown	Pink
Papaverine	80·6	87·5	Greenish-Yellow	Orange	Violet
Narcotine	73·0	78·1	Deep Blue	Orange	Violet
Atropine	50·0	80·6		Orange	Violet
Hyoscene	33·7	46·8		Red	Deep violet
Hyoscyamine	23·6	60·0		Orange	Violet
Scopolamine	32·5	43·7		Red	Deep violet
Strychnine	42·0	70·0		Orange	Violet
Brucine	30·0	58·1		Yellow	Deep blue
Aconitine	98·7	98·7		Orange	Yellow
Cocaine	62·5	75·0		Orange	Pink
Novocaine	36·8	57·0		Orange	Pink
Quinine	62·5	78·7	Blue*	Orange	Violet
Physostigmine	50·0	82·5	Violet	Red	Pink
Nicotine	16·8	26·2		Orange	Violet
Arecoline	26·8	40·6		Orange	Violet
Berberin	46·8	54·3	Yellow*	Yellow	Violet
Solanine	60·0	46·8			Pink
Flaxidil	3·75	4·37			Reddish Brown
Pathedine	83·7	96·2			Reddish
Spartine	9·37	27·5			Bluish
Cinchonin	48·1	77·5	Violet		Bluish black
Gelsemine	47·2	72·6			Brown

*Fluorescence.

The chromatograms were then suspended in a developing tank containing the solvent mixture. Ascending technique for the development was used in saturated chromatographic tank. The chromatograms were taken out after 6 hours when the solvent travelled upto 16 cms. After marking the solvent front, they were dried at room temperature and the spots of various alkaloids were observed under U. V. light (254 mμ) and marked. The spots on the chromatogram were then finally developed by spraying with the modified Dragendorff's reagent developed by Munier and Macheboeff[16] and with iodoplatinate reagent[17,18]. Most of the akaloids are well differentiated. Characteristic colours and colour changes are observed during or after spraying. Some alkaloids do not become visible until the reagent dries. The colours are also affected by the amount of alkaloids applied on the paper. The edge of the spots may have a different colour f om that ol the centre. Standard Rf. data for the 29 alkaloids studied were obtained using this procedure and recorded in Table 1.

It was observed that the salt buffered papers gave satisfactory clear cut separation of alkaloids. The amount of fat and protein extracted from the tissues alongwith alkaloids seems to have no effect on the clear resolation of the pure alkaloids.

Extraction of Biological Material

In a bottle or separating funnel add 10 to 15 ml. of chloroform to 1 ml. of alkalinized urine, blood or tissue slurry. Shake gently or agitated in case of bottles in a shaking machine for 20 minutes, aspirate the supernatant and wash the chloroform layer and the emulsion if any with an equal volume of 0·001 M Sodium hydroxide. Extract the washed organic layer of chloroform with three 20 ml. portions of 0·5 M Sulphuric acid and pool the aqueous acid extracts. Filter the residual chloroform, which contains the neutral fraction, through a dry filter paper and evaporate the filtrate to dryness on a water bath. Make the pooled aqueous acid extracts strongly alkaline and extract with three 25 ml. portions of chloroform. Filter the pooled chloroform extracts through dry filter paper and evaporate this fraction, which contains the alkaloids, to dryness. Take up the residue in 0·5 ml. of ethanol and spot aliquots of extract on the chromatograms. The diameter of the spot at the point of application should not exceed 4 to 5 mm. Normally, eight or nine spots can be conveniently applied on the paper. The chromatograms are developed as described earlier.

Isolation of the alkaloids from the chromatograms

The chromatograms after development are removed from the tank and air dried to remove the more volatile solvents and then kept in a hot case at 100°C for 10 min. The paper, while still warm is placed under ultraviolet light and the fluorescent or absorbent spots marked with a lead pencil. The chromatogram is placed on a sheet of glass and all but a 10 mm thin section of the paper is covered by another glass plate. The component on the 10 mm sections are visualized by spraying with modified Dragendorff's reagent or with iodoplatinate solution. The alkaloids shows up into a coloured spot. The corresponding alkaloids on the remainder of the chromatogram are isolated by cutting the paper with a razor blade above and below each compound. For elution, the cuttings are treated with 10 ml. of chloroform in a micro separating funnel and the latter is shaken for 5 minutes. The organic layer is evoporated on a water bath to obtain the alkaloid which can be subjected for identification by physical or chemical means.

The alkaloids from the chromatograms sprayed with Dragendorff's reagent or Iodoplatinate solution can also be recovered quantitatively. The coloured

spots of alkaloids on the paper are cut out and placed into a small micro separating funnel, to which are added 4 ml. of 0·5 N Sodium hydroxide and about 0·2 gm. of Sodium sulphite. The coloured alkaloid complex is completely destroyed, thus liberating the free alkaloid. About 10 ml. of chloroform are added to the separating funnel and the latter is shaken for 5 minutes to extract the basic salt. The aqueous layer is discarded and the solvent layer is extracted with 5 ml. of 0·5 N Sulphuric acid. The alkaloid sulphate in the aqueous phase can be obtained by evaporating the solution on a water bath.

Identification of alkaloids.

The isolated alkaloids in micro-quantities from the chromatograms needs identification. The toxicological chemist is expected to give positive identification to any alkaloidal substance that he isolates, and not merely to a selected few of classical forensic interest. Colour tests, while sometimes of great value as confirmatory tests, are not of universal application and suffer from the defects that micro-traces of impurities may give rise to misleading results. Physiological tests, using living animals, are, on the whole, less sensitive in microgram scale than chemical tests and can be carried out where facilities are available. Of all the techniques available for the identification of micro-quantities of alkaloids, the micro crystal tests are generally acknowledged to be the most delicate and the most reliable.

Several workers like Wormley[19], Fulton[20,21], Oliverio[22], Stephenson[23], White[24] and Wagenaar[25] developed micro crystalline tests by introducing new reagents. Tests have usually been carried out by placing a drop of the test solution on a microscope slide, adding a drop of the reagent, and examining under the low power of the microscope. These methods proved unsatisfactory when applied to micro quantities of alkaloids.

The technique described by Clarke[26] for the identification of micro quantities of alkaloids was found to be satisfactory and has been successfully applied in this work for preparing micro-crystals of alkaloids eluted from paper chromatograms.

Experimental

Microcrystal tests

The test is carried out with a micro drop whose volume is approximately 0·1 Cu.mm. The alkaloids were dissolved in 1 percent acetic acid or in 1 percent hydrochloric acid if insoluble in the former. The surface of this solution of the alkaloids is then touched with a micro-rod and the adhering drop of liquid transferred to a cover slip placed on the stand. A similar drop of reagent (*vide* list given in table II) is now added by means of another glass rod, and the drop stirred.

TABLE II
Reagents for microchemical tests

1.	Picric acid	5 percent solution
2.	Zinc chloride	5 percent solution
3.	Sodium carbonate	5 percent solution
4.	Lead iodide	5 percent solution
5.	Sodium phosphate	5 percent solution of Na_2HPO_4
6.	Gold chloride	5 percent solution
7.	Potassium chromate	5 percent solution
8.	Gold bromide	5g. $AuCl_3$+5g. NaBr in 100ml. of water
9.	Potassium tri-iodide	2g. I_2+4g. KI in 100ml. of water
10.	Potassium cadmium iodide	1g.CdI_2+2g.KI in 100ml. of water
11.	Potassium permanganate	2g KMnO$_4$ in 100ml of water + 5 drops of syrupy phosphoric acid.

The distance pieces of a pedestal slide are wetted with gum, and the cover-slip picked up by inverting the slide over it and pressing the distance pieces against its edges. The whole is now reinverted and examined under the low power of the microscope. The slide is examined for the crystals. The shape of the crystals is noted and photographs may be taken for purposes of record, as although many crystal formations are permanent, others disintegrate fairly quickly. The tests for a number of alkaloids are recorded in Table III.

TABLE III

Microcrystals of alkaloids

Alkaloids	Reagents	Crystals	Sensitivity μg.
Morphine	Potassium tri-iodide	Orange plates	0·05
Codeine	Potassium cadmium iodide	Rosettes of needles	0·05
Thebaine	Sodium carbonate	Rosettes	0·10
Apomorphine	Gold Bromide	Serratted crosses	0·10
Narceine	Lead iodide	Thread like needles	0 05
Papaverine	Zinc chloride	Bunches of plates	0·05
Narcotine	Sodium carbonate	Small rosettes	0·05
Atropine	Picric acid	Bunches of plates	0·25
Hyoscyamine	Picric acid	Rosettes of needles	0·25
Scopolamine	Gold bromide	Serrated yellow plates	0·10
Strychnine	Gold chloride	Feathery rosettes	0·10
Brucine	Potassium chromate	Bunches of rods	0·05
Aconitine	Potassium permanganate	Dense rosettes of rods	0·25
Cocaine	Gold chloride	Serrated needles	0 05
Quinine	Sodium phosphate	Needles	0·10
Physostigmine	Lead iodide	Serrated blades	0·25
Nicotine	Picric acid	Branching needles	0·05
Arecoline	Picric acid	Small dense rosettes	0·25

In table II is recorded one of the selected out of many available tests for alkaloids. They were found to be most suitable for use on the micro-quantities of alkaloids. The crystals formed from a test solution are compared with that of a control solution of practically similar concentraction from a known sample of the alkaloid salt.

Identification of an alkaloid by performing colour tests has been as old as the subject of toxicology. Bamford[27] worked out a scheme for the identification of alkaloids by colour reactions. The method works well with pure sample of alkaloids but do not give always satisfactory results with alkaloids obtained from the tissue. This problem has too been solved by obtaining the pure alkaloid after separation and elution from the chromatogram described earlier. For want of space the colour test technique has not been incorporated in this text and will form the subject matter for future communication.

Acknowledgement

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Application of an integral equation involving generalized Laguerre polynomial in the evaluation of certain integrals and in the problem of mortality of equipment*

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1. Introduction

Recently, the author[1] has obtained the following theorems with $\lambda = 0$.

Theorem I. If k and λ are complex numbers and α is a non-negative integer; $f^{\alpha+1}(t)$ is continuous for $0 \leqslant t < a < \infty$ and $f(0) = f'(0) = \ldots = f^{\alpha}(0) = 0$ then the integral equation

$$(1 \cdot 1) \qquad \int_0^t e^{\lambda(t-u)} (t-u)^\alpha L_n^\alpha [k(t-u)] g(u) \, du = f(t)$$

has the solution

$$(1 \cdot 2) \qquad g(t) = \frac{(n)!}{(\alpha+n)!} \left(\frac{d}{dt} - \lambda \right)^{\alpha+1} f(t) +$$

$$\frac{(n)!}{(\alpha+n)!} \int_0^t \left[\left(\frac{d}{du} - \lambda \right)^{\alpha+1} f(u) \right] \psi(t-u) \, du \quad \text{for} \quad 0 \leqslant t < a$$

where

$$(1 \cdot 3) \qquad \psi(t) = e^{(k+\lambda)t} \sum_{r=1}^n {}^n c_r \frac{k^r \, t^{r-1}}{(r-1)!}$$

$$= k \, n \, e^{(k+\lambda)t} \, {}_1F_1 (-n+1 \, ; \, 2 \, ; -kt) :$$

$$= k \, e^{(k+\lambda)t} \, L'_{n-1} (-kt).$$

Theorem II. If k and λ are complex numbers and α is a non-negative integer; $f^{2\alpha+2}(t)$ is continuous for $0 \leqslant t < a < \infty$ and $f(0) = f'(0) = \ldots = f^{2\alpha+1}(0) = 0$ then the integral equation $(1 \cdot 1)$ has the solution

$$(1 \cdot 4) \qquad g(t) = \left[\frac{(n)!}{(\alpha+n)!} \right]^2 \int_0^t [\{ (D - \lambda) (D - \lambda - k) \}^{\alpha+1} f(u)] \times$$

$$(t - u)^\alpha \, e^{(\lambda+k) (t-u)} \, L_n^\alpha [k(u-t)] \, du \text{ for } 0 \leqslant t < a, \quad D \equiv \frac{d}{du}$$

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*This is a part of my Thesis approved for the degree of Ph.D.

In the section 3 of this paper we evaluate two integrals involving generalized Laguerre polynomial by the application of theorems I and II. In the section 4 we discuss the application of a particular case of the integral equation (1·1) in the problem of mortality of equipment.

2. *Results required in the proof :*

Representing Laplace transform $\int_0^\infty e^{-pt} f(t)\ dt$ of a function $f(t)$ by $L\{f(t)\}$, we have from [2, p. 131, p. 144, p. 175].

$$(2\cdot1) \qquad L\left\{ \int_0^t f_1(u)\, f_2(t-u)\ du \right\} = L\{f_1(t)\}.\ L\{f_2(t)\},$$

$$(2\cdot2) \qquad L\{ t^{\nu-1}\, e^{-\alpha t} \} = \frac{\Gamma(\nu)}{(p+\alpha)^\nu}\ ,\ Re\ \nu > 0,\ Re\ p > Re\ \alpha,$$

$$(2\cdot3) \qquad L\left\{ t^\alpha\, e^{\lambda t}\, L_n^\alpha\ (kt) \right\} = \frac{\Gamma(\alpha+n+1)}{(n)\,!} \cdot \frac{(p-k-\lambda)^n}{(p-\lambda)^{\alpha+n+1}}\,,$$
$$Re\ \alpha > -1,\ Re\ (p-\lambda) > 0.$$

3. *Evaluation of integrals.*

$$(3\cdot1) \qquad \int_0^t \left[\left(\frac{d}{du} - \lambda \right)^{\alpha+1} (u^{\alpha+n}\, e^{\lambda u}) \right] \psi(t-u)\ du$$

$$= \frac{(\alpha+n)\,!}{(n-1)\,!}\ t^{n-1}\, e^{(k+\lambda)t} - \left(\frac{d}{dt} - \lambda \right)^{\alpha+1} \left(t^{\alpha+n}\, e^{\lambda t} \right)$$

where α is a non-negative integer, n is a positive integer such that $n \geqslant 1$ and $\psi(t)$ is given by (1·3).

$$(3\cdot2) \qquad \int_0^t \left[\left\{ \left(\frac{d}{du} - \lambda \right) \left(\frac{d}{du} - \lambda - k \right) \right\}^{\alpha+1} (u^{\alpha+n}\, e^{\lambda u}) \right] \times$$

$$(t-u)^\alpha\, e^{(\lambda+k)(t-u)}\ L_n^\alpha\, [\, k(u-t)\,]\ du = \left[\frac{(\alpha+n)!}{(n)\,!} \right]^2 n\ t^{n-1}\, e^{(\lambda+k)t}$$

where α is a non-negative integer and n is a positive integer such that $n \geqslant \alpha + 2$.
Proof :

We have the following identity :

$$(3\cdot3) \qquad \left[\frac{(\alpha+n)\,!}{(n)\,!} \cdot \frac{(p-\lambda-k)^n}{(p-\lambda)^{\alpha+n+1}} \right] \left[\frac{(n-1)\,!}{(p-\lambda-k)^n} \right] = \frac{1}{n} \left[\frac{(\alpha+n)\,!}{(p-\lambda)^{\alpha+n+1}} \right].$$

Applying the results (2·3) and (2·2) to (3·3), we get

$$(3\cdot4) \qquad L\{ t^\alpha\, e^{\lambda t}\, L_n^\alpha\ (kt) \}.\ L\{ t^{n-1}\, e^{(\lambda+k)t} \} = \frac{1}{n}\ .\ L\{ t^{\alpha+n}\, e^{\lambda t} \}$$

Applying (2·1) to (3·4), we get

$$(3\cdot5) \qquad \int_0^t (t-u)^\alpha\, e^{\lambda(t-u)}\ L_n^\alpha\ [k\,(t-u)]\ u^{n-1}\, e^{(\lambda+k)u}\ du = \frac{t^{\alpha+n}\, e^{\lambda t}}{n}$$

[178]

Comparing the result (3·5) with the integral equation (1·1), we find that

(3·6)
$$f(t) = \frac{t^{a+n} e^{\lambda t}}{n} \quad \text{and}$$

(3·7)
$$g(t) = t^{n-1} e^{(\lambda+k)t}$$

By theorem I, the integral equation (1·1) has the solution (1·2) provided $f(0) = f'(0) = \ldots \ldots = f^a(0) = 0$.

Taking $f(t)$ given by (3·6), we find that

$$f(0) = f'(0) = \ldots \ldots = f^{a+n-1}(0) = 0.$$

Therefore we must have $a \leqslant \alpha + n - 1$ or $n \geqslant 1$.

Substituting the values of $f(t)$ and $g(t)$ from (3·6) and (3·7) respectively into (1·2) and simplifying, we get (3·1).

By theorem II, the integral equation (1·1) has the solution (1·4) provided $f(0) = f'(0) = \ldots \ldots = f^{2a+1}(0) = 0$.

We again take $f(t)$ and $g(t)$ which are given by (3·6) and (3·7).

Here we must have $2a + 1 \leqslant a + n-1$ or $n \geqslant \alpha + 2$.

Substituting the values of $f(t)$ and $g(t)$ from (3·6) and (3·7) into (1·4) and simplifying, we get (3·2).

4. In the article "mortality of equipment" [3, pp. 104–105], we find the integral equation

(4·1)
$$F(t) = F(0) H(t) + \int_0^t H(t-u) R'(u) \, du$$

where $F(t)$ denotes the number of pieces of equipment on hand at time t, $H(t)$ is a function that determines the surviving equipment after t units of time and $R'(t)$ is the rate of replacements. We also find $H(0) = 1$ necessarily.

If $F(t)$ and $H(t)$ are known, then the equation (4·1) is an integral equation of convolution type in $R'(t)$. Its solution gives the formula by which replacements must be made.

Here we take

(4·2)
$$F(t) = \frac{t^n e^{-\beta t}}{n} \quad \text{and}$$

(4·3)
$$H(t) = e^{-\beta t} L_n(-kt)$$

where n is a positive integer, β is a positive or negative real number and k is a positive real number, and $L_n(-kt)$ is a Laguerre polynomial; and determine $R'(t)$. We have $F(0) = 0$ and $H(0) = 1$.

Therefore, from (4·1) we get

(4·4)
$$\int_0^t e^{-\beta(t-u)} L_n[k(u-t)] R'(u) \, du - \frac{t^n e^{-\beta t}}{n}$$

[179]

which is a particular case of the integral equation $(1\cdot1)$ when we take $-k$ for k,

$$a = 0, \lambda = -\beta \quad \text{and} \quad f(t) = \frac{t^n \, e^{-\beta t}}{n} \, .$$

Applying $(2\cdot1)$, $(2\cdot2)$ and $(2\cdot3)$ to $(4\cdot4)$, we get

$$(4\cdot5) \qquad L\{ R'(t) \} = \frac{(n-1)!}{(p+k+\beta)^n}$$

From $(2\cdot2)$ and $(4\cdot5)$ we get

$$(4\cdot6) \qquad R'(t) = t^{n-1} \, e^{-(k+\beta)t}$$

Taking β to be positive, n can be suitably chosen so that the function $F(t)$ given by $(4\cdot2)$ decreases as t increases. In this case the function $F(t)$ may be suitable for a civil engineering construction-project in which major amount of equipment is required to begin the project while as the project progresses, the amount of equipment required, is lesser and lesser.

Taking β to be negative, the function $F(t)$ increases as t increases. In this case $F(t)$ may be suitable for a project like production factory or a power plant in which lesser amount of equipment is required to start the project and as the project progresses, more equipment can be added out of its own earnings.

By taking $H(t)$ given by $(4\cdot3)$, we cover a large number of functions because a function satisfying certain conditions, can be expanded into an infinite series of Laguerre polynomials [4, p. 88].

Acknowledgement

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Reduction of Triple integral equations to Dual finite series equations

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1. Introduction

In some potential problems different conditions hold over three different parts of the same boundary. In many cases these problems can be reduced to the solution of triple integral equations.

The solution of the triple integral equations

$$(1\cdot1) \qquad \int_0^\infty \phi(u)\, J_\nu(r\,u)\, du = f(r),\ 0 < r < a$$

$$(1\cdot2) \qquad \int_0^\infty u^{2p}\, \phi(u)\, J_\nu(r\,u)\, du = g(r),\ a < r < b$$

$$(1\cdot3) \qquad \int_0^\infty \phi(u)\, J_\nu(r\,u)\, du = 0,\ b < r < \infty$$

has been discussed by Tranter [1] who has reduced these equations to dual infinite series equations

$$(1\cdot4) \qquad \sum_{n=1}^\infty \frac{\Gamma(n+\nu)}{\Gamma(n+p)}\, C_n\, {}_2F_1\left(-n+1, n+\nu+p;\ \nu+1;\ \frac{r^2}{b^2}\right)$$
$$= 2^p\, \Gamma(\nu+1)\, b^{\nu+1-p}\, r^{-\nu}\left(1 - \frac{r^2}{b^2}\right)^{-p} f(r),\ 0 < r < a,$$

$$(1\cdot5) \qquad \sum_{n=1}^\infty \frac{\Gamma(n+\nu+p)}{\Gamma(n)}\, C_n\, {}_2F_1\left(n+1, n+\nu+p;\ \nu+1;\ \frac{r^2}{b^2}\right)$$
$$= 2^{-p}\, \Gamma(\nu+1)\, b^{\nu+1+p}\, r^{-\nu}\, g(r),\ a < r < b$$

under the conditions that, when $p = \frac{1}{2}$, $\nu > -1$; $p = -\frac{1}{2}$, $\nu > -\frac{1}{2}$.

In this paper we consider the triple integral equations

$$(1\cdot6) \qquad \int_0^\infty \phi(u)\, J_\nu(r\,u)\, du = 0,\ 0 < r < a$$

$$(1\cdot7) \qquad \int_0^\infty u^{-2p+2}\, \phi(u)\, J_\nu(r\,u)\, du = f(r),\ a < r < b$$

[181]

$$(1\cdot8) \qquad \int_0^\infty \phi(u)\, J_\nu(r\,u)\, du = g(r),\ b < r < \infty$$

and reduce these equations to dual finite series equations.

2. Results required in the proof:

We have from [2, p. 35, p. 83] and [3, p. 60).

$$(2\cdot1) \quad \int_0^\infty u^{-\lambda} J_\mu(a\,u)\, J_\nu(b\,u)\, du = \frac{b^\nu\, \Gamma\!\left(\dfrac{\mu + \nu - \lambda + 1}{2}\right)}{2^\lambda\, a^{\nu-\lambda+1}\, \Gamma[\nu + 1]\, \Gamma\!\left(\dfrac{\lambda + \mu - \nu + 1}{2}\right)} \times$$

$$\qquad {}_2F_1\left(\frac{\mu + \nu - \lambda + 1}{2}\,,\ \frac{\nu - \lambda - \mu + 1}{2}\ ;\ \nu + 1\ ;\ \frac{b^2}{a^2}\right),\ 0 < b < a$$

provided $Re\,(\mu + \nu - \lambda + 1) > 0$, $Re\,\lambda > -1$, a and b are real.

$$(2\cdot2) \quad \int_0^\infty u^{-\lambda} J_\mu(a\,u)\, J_\nu(b\,u)\, du = \frac{a^\mu\, \Gamma\!\left(\dfrac{\nu + \mu - \lambda + 1}{2}\right)}{2^\lambda\, b^{\mu-\lambda+1}\, \Gamma(\mu + 1)\, \Gamma\!\left(\dfrac{\lambda + \nu - \mu + 1}{2}\right)} \times$$

$$\qquad {}_2F_1\left(\frac{\nu + \mu - \lambda + 1}{2}\,,\ \frac{\mu - \lambda - \nu + 1}{2}\ ;\ \mu + 1\ ;\ \frac{a^2}{b^2}\right),\ 0 < a < b$$

provided $\qquad Re\,(\mu + \nu - \lambda + 1) > 0$, $\quad Re\,\lambda > -1$, a and b are real.

$$(2\cdot3) \qquad F_m(\alpha, \beta, x) = {}_2F_1\,(-m, \alpha + m\ ;\ \beta\ ;\ x)$$

where $F_m\,(\alpha, \beta, x)$ is a Jacobi polynomial.

$$(2\cdot4) \qquad {}_2F_1\,(a, b\ ;\ c\ ,\ z) = (1 - z)^{c-a-b}\, {}_2F_1\,(c - a, c - b\ ;\ c\ ;\ z).$$

3. Solution of the equations (1·6), (1·7), (1·8):

Substituting $\lambda = 1 - p$, $\mu = \nu - 2n + p - 2$; n being a positive integer, and $b = r$ in (2·1), we get

$$(3\cdot1) \qquad \int_0^\infty u^{p-1} J_{\nu-2n+p-2}\,(a\,u)\, J_\nu(r\,u)\, du = 0,\ 0 < r < a$$

provided

$$(3\cdot2) \qquad\qquad\qquad p < 2,$$
$$(3\cdot3) \qquad\qquad\qquad n < \nu + p - 1.$$

Substituting $\lambda = p - 1$, $\mu = \nu - 2n + p - 2$, n being a positive integer, and $b = r$ in (2·2), we get

[182]

$(3\cdot4)$ $\displaystyle\int_0^\infty u^{1-p}\, J_{\nu-2n+p-2}\,(a\,u)\, J_\nu(r\,u)\, du = \frac{2^{1-p}\,\Gamma(\nu-n)\,a^{\nu-2n+p-2}}{\Gamma(\nu-2n+p-1)\,\Gamma(n+1)} \times$

$$r^{2n-\nu}\ {}_2F_1\left(\nu-n,\ -n\ ;\ \nu-2n+p-1\ ;\frac{a^2}{r^2}\right),\ 0<a<r$$

provided

$(3\cdot5)$ $\qquad\qquad\qquad\qquad p>0,$

$(3\cdot6)$ $\qquad\qquad\qquad\qquad n<\nu.$

Substituting $\lambda=1-p,\ \mu=\nu-2n+p-2,\ n$ being a positive integer, and $b=r$ in $(2\cdot2)$, we get

$(3\cdot7)$ $\displaystyle\int_0^\infty u^{p-1}\, J_{\nu-2n+p-2}\,(a\,u)\, J_\nu(r\,u)\, du =\frac{2^{p-1}\,\Gamma(\nu-n+p-1)}{\Gamma(\nu-2n+p-1)\,\Gamma(n-p+2)} \times$

$$a^{\nu-2n+p-2}\ r^{2n-\nu-2p+2}\ {}_2F_1\left(\nu-n+p-1,\ -n+p-1\ ;\ \nu-2n+p-1\ ;\frac{a^2}{r^2}\right),$$
$$0<a<r$$

provided

$(3\cdot8)$ $\qquad\qquad\qquad\qquad p<2,$

$(3\cdot9)$ $\qquad\qquad\qquad\qquad n<\nu+p-1.$

Applying $(2\cdot4)$ to $3\cdot7)$, we get

$(3\cdot10)$ $\displaystyle\int_0^\infty u^{p-1}\, J_{\nu-2n+p-2}\,(a\,u)\, J_\nu(r\,u)\, du =\frac{2^{p-1}\,\Gamma(\nu-n+p-1)}{\Gamma(\nu-2n+p-1)\,\Gamma(n-p+2)}$

$$a^{\nu-2n+p-2}\ r^{2n-\nu-2p+2}\left(1-\frac{a^2}{r^2}\right)^{1-p}\times$$

$${}_2F_1\left(-n,\ \nu-n\ ;\ \nu-2n+p-1\ ;\frac{a^2}{r^2}\right),\ 0<a<r$$

combining the conditions $(3\cdot2)$, $(3\cdot5)$ and $(3\cdot8)$, we get

$(3\cdot11)$ $\qquad\qquad\qquad 0<p<2.$

combining the conditions $(3\cdot3)$, $(3\cdot6)$ and $(3\cdot9)$, we get

$(3\cdot12)$ $\quad n$ can be any positive integer which is less than the smaller of the values of ν and $\nu+p-1$.

If we take

$(3\cdot13)$ $\qquad\qquad\qquad \phi(u)=u^{p-1}\displaystyle\sum_{n=1}^{n^1} c_n\, J_{\nu-2n+p-2}\,(a\,u)$

where n' is the integral part in the smaller of the values of ν and $\nu+p-1$, the equation $(1\cdot6)$ is automatically satisfied because of $(3\cdot1)$ and $(3\cdot12)$.

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Substituting the expression for $\phi(u)$ from (3·13) into the equation (1·7) and using (3·4), (3·12) and (2·3), we get

$$(3·14) \qquad \sum_{n=1}^{n'} c_n \frac{a^{-2n}\, r^{2n}\, \Gamma(\nu - n)}{\Gamma(\nu - 2n + p - 1)\, \Gamma(n + 1)}\, F_n\left(\nu - 2n,\ \nu - 2n + p - 1,\ \frac{a^2}{r^2}\right)$$

$$= 2^{p-1}\, a^{-\nu-p+2}\, r^\nu\, f(r),\ a < r < b.$$

Substituting the same expression for $\phi(u)$ into the equation (1·8) and using (3·10), (3·12) and (2·3), we get

$$(3·15) \qquad \sum_{n=1}^{n'} c_n \frac{a^{-2n}\, r^{2n}\, \Gamma(\nu - n + p - 1)}{\Gamma(\nu - 2n + p - 1)\, \Gamma(n - p + 2)}\, F_n\left(\nu - 2n,\ \nu - 2n + p - 1,\ \frac{a^2}{r^2}\right)$$

$$= 2^{-p+1}\, a^{-\nu-p+2}\, r^{\nu+2p-2}\left(1 - \frac{a^2}{r^2}\right)^{p-1} g(r),\ b < r < \infty.$$

The coefficients c_n are to be determined from the dual finite series equations (3·14), (3·15). Once these coefficients are determined, (3·13) gives the solution of the triple integral equations (1·6), (1·7), (1·8) under the conditions (3·11) and (3·12).

Acknowledgement

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Effect of Transverse Magnetic Field on an Incompressible Conducting Laminar Boundary Layer with Pressure Gradient

By

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Abstract

A modified Kármán–Pohlhausen approach is employed to solve boundary layer flows of an electrically conducting fluid on a body of arbitrary shape with arbitrarily prescribed transverse magnetic field. Analytical solutions are obtained for a linearly varying outer velocity distribution on a flat plate and for various values of the magnetic parameter $\wedge_m = \dfrac{\delta^2 \, \sigma \, B^2}{\rho \, \nu}$. For $\wedge_m = 0$, Howarth type flow results are compared with the available exact series solution. It is found that the agreement is good. Effect of the magnetic field on the various characteristics of the flow is calculated.

Nomenclatute

x, y	=	Cartesian coordinates along and perpendicular to the body surface.
u, v	=	Longitudinal and transverse velocity components.
p, ρ, μ, σ	=	Pressure, density, viscosity and electrical conductivity of the fluid respectively.
ν	=	Kinetic viscosity.
$\delta(x)$	=	Boundary layer thickness.
Re	=	$\dfrac{\rho\infty \, U\infty \, L}{\mu\infty}$ Reynolds number.
$\tau(x)$	=	Wall shear stress.
$U(x)$	=	Free stream velocity.
$B(x)$	=	Magnetic induction vector.
$\overrightarrow{J}$	=	Current density vector.
$\wedge_m$	=	$\dfrac{\delta^2 \, \sigma \, B^2}{\rho\nu}$ = Magnetic parameter.
$\wedge_p$	=	$\dfrac{\delta^2 \, U'}{\nu}$ = Shape factor. Here, ' denotes differentiation with respect to x.
$\wedge$	=	$(\wedge_m + \wedge_p)$.
x_s	=	Point of separation.
R_m	=	$\mu\infty \, \sigma \, U\infty \, L$ Magnetic Reynold's number.

[185]

Introduction

A modified Kármán–Pohlhausen method as suggested by Torda[1] and later used by Jain and Bhatnagar[2,3] is used to calculate the flow characteristics of an electrically conducting fluid on a body with arbitrary pressure gradient and in the presence of a transverse magnetic field. An equation determining the variation of the shape parameter $\wedge_p$ and the magnetic parameter $\wedge_m$ with the longitudinal distance on the body is derived, which can be solved by numerical or graphical procedures for a given pressure distribution on the body and a given magnetic field. However, in certain cases of interest, analytical solution can be derived. We have obtained closed form solutions for a linearly varying outer velocity distribution and carried out a detailed investigation of the flow with and without magnetic field. Comparison is made with the classical Pohlhausen method[1] and with the series solution[5] for the Howarth type flow $\overline{U} = 1 - \overline{x}$ in the absence of magnetic field. It is observed that the agreement with the exact solution is good. The point of separation is $\overline{x}_s = 0.1116$ instead of $\overline{x}_s = 0.120$ given by the series solution. It may further be remarked that contrary to our expectation, the present procedure spoils the agreement for the accelerated flow[2] and for flow on flat plate with no pressure gradient[6]. Besides, we find that the present scheme fails to give any physically plausible results for flow near the stagnation point of a blunt body. This forfeits the usual notion that more boundary conditions on the surface than at the edge give better results in all cases.

Computations are also carried out for various values of the magnetic parameter $\wedge_m$. It is found that as the strength of the magnetic field increases the boundary layer thickness decreases, the skin-friction $\tau(x)$ increases. Consequently the point of separation shifts away from the leading edge, the displacement thickness $\overline{\delta}^*$ and the momentum thickness, $\overline{\theta}$ decreases the decrement in $\overline{\delta}^*$ is more marked than the decrement in $\overline{\theta}$.

Previous work on boundary layer flows with transverse magnetic field is carried out by Maciulaitis and Loeffler[7], Moffat[8], Dhanak[9] but they essentially solved problems related to boundary layer flows in channels. Heiser and Bornhorst[10] solved a similar problem by the modified Pohlhausen method, but the entire computational work is carried out numerically on digital computer. Besides, their profile does not satisfy the prescribed conditions at the edge of the boundary layer. Hence, their entire results are of limited validity.

The main assumptions underlying the procedure are the following :

(*i*) The density, conductivity and viscosity are each constant throughout the flow.

(*ii*) Displacement current and Hall current effects are negligible.

(*iii*) The free stream velocity and magnetic field (taken to be in y-direction) are externally controlled and known.

(*iv*) The electric field is (−UB) in the frame of reference of the boundary.

(*v*) Magnetic Reynolds number

$$\dot{R}_m = \frac{\text{Induced magnetic field}}{\text{Applied magnetic field}}$$

[186]

is supposed to be small compared with unity so that induced magnetic field can be neglected[11].

2. Equations of motion

The equations determining the steady, two-dimensional, laminar, incompressible boundary layer flow of an electrically conducting flow with transvese magnetic field are the following :

Equation of continuity :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2 \cdot 1)$$

Momentum equation :

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U(x)\ U'(x) + v\ \frac{\partial^2 u}{\partial y^2} + (\vec{J} \times \vec{B})_x \qquad (2 \cdot 2)$$

Boundary conditions are :

$$\left. \begin{array}{llll} y = 0, & u = 0, & v = 0 \\ y \to \infty, & u \to U \end{array} \right\} \qquad (2\ 3)$$

Here x, y are the distances measured along and perpendicular to the body surface and u, v are the components of the velocity in the corresponding directions. $U x)$ and $B'(x)$ are the prescribed outer velocity and the variable magnetic field.

With the assumptions stated in the introduction, equation (2·2) reduces to the following form :

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = U(x)\ U'(x) + v\ \frac{\partial^2 u}{\partial y^2} + \frac{\sigma B^2\ (U - u)}{\rho} \qquad (2 \cdot 4)$$

A first integral of the equation (2·4) with the boundary conditions (2·3) with the help of (2·1) can be obtained in the following form

$$\frac{d}{dx}\ \int_0^{\delta(x)} u^2 dy - U(x)\ \int_0^{\delta(x)} u\ dy$$

$$= \delta(x)\ U(x)\ U'(x) - v \left(\frac{\partial u}{\partial y} \right)_0 + \frac{\sigma B^2}{\rho} \int_0^{\delta(x)}\ (U - u)\ dy \qquad (2 \cdot 5)$$

3 Method of Solution

We solve (2·5) by assuming a fourth degree polynomial expression for velocity profile as follows :

$$\frac{u}{U} = a\ \eta + b\ \eta^2 + c\ \eta^3 + d\ \eta^4 \qquad (3 \cdot 1)$$

where $\qquad \eta = \frac{y}{\delta(x)}$.

The coefficients a, b, c, d are evaluated by using the following conditions :

$$\eta = 0, \qquad u = 0, \qquad v = 0 \qquad (3 \cdot 2a)$$

[187]

$$\eta = 0 \quad \frac{\partial^2}{\partial \eta^2}\left(\frac{u}{U}\right) = -\frac{\delta^2}{\nu}\left\{\frac{dU}{dx} + \frac{\sigma B^2}{\rho}\right\} \tag{3.2b}$$

$$\eta = 0 \quad \frac{\partial^3}{\partial \eta^3}\left(\frac{u}{U}\right) = \frac{\delta^2}{\rho \nu}\frac{\sigma B^2}{\rho}\frac{\partial}{\partial \eta}\left(\frac{u}{U}\right) \tag{3.2c}$$

$$\eta = 1 \quad \frac{u}{U} = 1, \frac{\partial}{\partial \eta}\left(\frac{u}{U}\right) = 0 \tag{3.2d}$$

Boundary condition (3.2c) is different from the one we use in the usual Pohlhausen method.

The coefficients are given as

$$a = \frac{6(4 + \Lambda)}{18 + \Lambda m}, \quad b = -\frac{\Lambda}{2},$$

$$c = \frac{\Lambda m}{18 + \Lambda m}(4 + \Lambda), \quad d = \frac{6 \Lambda p - \Lambda \Lambda m - 12}{2(18 + \Lambda m)} \tag{3.3}$$

Substituting (3.1) with (3.3) in the integrated momentum equation (2.5), we get,

$$\frac{\delta}{\rho \nu}\frac{\sigma B}{\ }\frac{dB}{dx} = \frac{2520}{U}\frac{D^3 \nu}{\delta^3 K_4}\left[\Lambda p\left(1 - \frac{K_2}{2520 D^2}\right) + \frac{\Lambda m}{60 D} K_6 - \frac{6(4 + \Lambda)}{D}\right]$$

$$- \frac{(DK_1 - K_3)}{\delta^2 K_4}\delta' - \frac{\delta}{K_4 \nu} K_5 U'' \tag{3.4}$$

where

$$K_1 = (-116928 + 1584 \Lambda p + 1224 \Lambda p^2)$$
$$+ \Lambda m(-12528 + 1536 \Lambda p - 66 \Lambda p^2)$$
$$+ \Lambda m^2(24 - 120 \Lambda p + \Lambda p^2) + \Lambda m^3(-54 + 2\Lambda p) + \Lambda m^4 \tag{3.4a}$$

$$K_2 = (256032 + 2448 \Lambda p^2 + 30384 \Lambda p)$$
$$+ \Lambda m(47520 + 3828 \Lambda p - 132 \Lambda p^2)$$
$$+ \Lambda m^2(1812 - 282 \Lambda p + 2\Lambda p^2) + \Lambda m^3(150 + 4 \Lambda p) + 2 \Lambda m^4 \tag{3.4b}$$

$$K_3 = (57024 \Lambda p + 88128 \Lambda p^2) + \Lambda m(16704 + 107424 \Lambda p - 7128 \Lambda p^2)$$
$$+ \Lambda m^2(29784 - 12960 \Lambda p + 12 \Lambda p^2) + \Lambda m^2(26784 - 12960 \Lambda p + 12 \Lambda p^2)$$
$$+ \Lambda m^2(-5832 + 48 \Lambda p + 4 \Lambda p^2) + \Lambda m^4(36 + 8\Lambda p) + 4 \Lambda m^5 \tag{3.4c}$$

$$K_4 = (1670 + 48960 \Lambda p - 7272 \Lambda p^2) + \Lambda m(26784 - 11712 \Lambda p$$
$$- 204 \Lambda p^2) + \Lambda m^2(-5832 + 216 \Lambda p) + \Lambda m^3(36 + 4 \Lambda p) + 4\Lambda m^4 \tag{3.4d}$$

$$K_5 = (28512 + 44064 \Lambda p) + \Lambda m(29232 + 72 \Lambda p) + \Lambda m^2(-624 - 96 \Lambda p)$$
$$+ \Lambda m^3(-84 + 2 \Lambda p) + 2 \Lambda m^4 \tag{3.4e}$$

$$K_6 = (432 - 36 \Lambda p) - \Lambda m(54 - \Lambda p) + \Lambda m^2 \tag{3.4f}$$

$$D = 18 + \Lambda m \tag{3.5g}$$

[188]

The equation $(3\cdot4)$ gives the variation of the boundary layer thickness (δ) for the prescribed values of $U(x)$ and $B(x)$ against the longitudinal distance x of the body. In general this equation can be solved by numerical or graphical means for any given $U(x)$ and $B(x)$. However, for a few special cases of interest, this equation can be integrated to give analytical solutions. We shall discuss them in the following section.

4. Special Cases

Let us consider the following outer velocity distribution, linearly varying with the distance x.

$$U(x) = U_0 + U_1 x \tag{4.1}$$

where U_0 and U_1 are constants. By prescribing different values to U_0 and U_1, we can get accelerating flows and decelerating flows. In particular, we obtain Howarth type flow for $U_0 = 1$ and $U_1 = -1$.

With $(4\cdot1)$, equation $(3\cdot4)$ can be written as

$$\frac{(D\,K_1 + K_3) - \wedge_m K_4}{D[\,\wedge_p\,(2520\,D^2 - K_2) + 42D\,\wedge_m + K_6 - 6(4 + \wedge)\,2520D]}\,\frac{d\wedge_p}{dx}$$

$$= \frac{2\,U'(x)}{U(x)} \tag{4.2}$$

We integrate $(4\cdot2)$ for $\wedge_m = 0, 1, 2$ and evaluate the constants of integration from the condition,

$$x = 0, \qquad \wedge_p = 0 \tag{4.3}$$

We record below the velocity profiles and the integrated forms of the equation $(4\cdot2)$ when $\wedge_m = 0, 1$ and 2.

(i) $\wedge_m = 0$.

Velocity profile :

$$\frac{u}{U} = \frac{4 + \wedge_p}{3}\,\eta - \frac{\wedge}{2}\,\eta^2 + \frac{\wedge_p - 2}{6}\,\eta^4 \tag{4.4}$$

Point of separation

$$\wedge_p = -4 \tag{4.5}$$

Integral of $(4\cdot2)$ with $(4\cdot3)$ is

$$3\cdot193875\,l_n\,(\wedge_p + 19\cdot5844) + 0\cdot903062\,l_n\,(\wedge_p^2 - 7\cdot172635\,\wedge_p + 22\cdot707053)$$

$$- 0\cdot670427\,\tan^{-1}\,\frac{(\wedge_p - 3\cdot586317)}{3\cdot137731} = -4\,l_n\,\frac{U(x)}{U_0} + 12\cdot890365 \tag{4.6}$$

(ii) $\wedge_m = 1$.

Volocity Profile :

$$\frac{u}{U} = \frac{6(5 + \wedge_p)}{19}\,\eta - \frac{(\wedge_p + 1)}{2}\,\eta^2 + \frac{(5 + \wedge_p)}{19}\,\eta^3 + \frac{(5\,\wedge_p - 13)}{38}\,\eta^4 \tag{4.7}$$

[189]

Point of separation :
$$\wedge_p = - 5 \tag{4·8}$$

Integral of equation (4·2) with (4·3) is

$$1·65316 \, l_n(\wedge_p + 21·50078) + 0·42357 \, l_n(\wedge_p{}^2 - 6·86144 \, \wedge_p + 22·72303)$$

$$- 0·41684 \, \tan^{-1} \frac{(\wedge_p - 3·43072)}{3·30956} = - 2 \, l_n \frac{U(x)}{U_0} + 6·72980 \tag{4·9}$$

(iii) $\wedge_m = 2$

Velocity Profile :

$$\frac{u}{U} = \frac{6(6 + \wedge_p)}{20} \, \eta - \frac{(\wedge_p + 2)}{2} \eta^2 + \frac{2(6 + \wedge_p)}{20} \eta^3 + \frac{(4 \, \wedge_p - 16)}{40} \eta^4 \tag{4·10}$$

Point of separation :

$$\wedge_p = - 6 \tag{4·11}$$

Integral of equation (4·2) with (4·3) is

$$1·47468 \, l_n \, (\wedge_p + 23·54080)$$

$$+ 0·43432 \, l_n(\wedge_p{}^2 - 6·68679 \, \wedge_p + 24·50728)$$

$$- 0·36784 \, \tan^{-1} \frac{(\wedge_p - 3·34339)}{3·65048} = - 2 \, l_n \frac{U(x)}{U_0} + 6·32025 \tag{4·12}$$

(iv) To compare and to point out clearly the effect of the additional boundary condition (3·2b) on the surface in the case of no magnetic field, we have also evaluated the fourth degree profile (3·1) using the boundary conditions

$$\eta = 0 \qquad u = 0 \qquad v = 0 \\ \left. \begin{array}{l} \frac{\partial^2 u}{\partial \eta^2} \left(\frac{u}{U} \right) = - \wedge_p \end{array} \right\} \\ \eta = 1 \qquad \frac{u}{U} = 1 \qquad \frac{\partial}{\partial \eta} \left(\frac{u}{U} \right) = 0 \tag{4·13}$$

Thus we get,

$$\frac{u}{U} = \frac{12 + \wedge_p}{6} - \frac{\wedge_p}{2} \, \eta^2 - \frac{(\wedge_p + 4)}{2} \, \eta^3 + \frac{(6 - \wedge_p)}{6} \, \eta^4 \tag{4·14}$$

Point of separation

$$\wedge_p = - 12 \tag{4·15}$$

Using (4·14) and (2·5) we get the integral of the resulting equation in the following form :

$$0·179976 \, l_n \, (7·0523 - \wedge_p)$$

$$+ 0·535013 \, l_n \, (1286·39 - 54·4525 \, \wedge_p - \wedge_p{}^2)$$

$$+ 0·267970 \, l_n \frac{72·2558 + \wedge_p}{17·8032 - \wedge_p} = - l_n \frac{U(x)}{U_0} + 4·557420 \tag{4·16}$$

In this note we consider only the case of decelerating flows given by

$$\overline{U}(x) = 1 - \overline{x} \tag{4.17}$$

where

$$\overline{U}(x) = \frac{U(x)}{U_\infty}, \quad \overline{x} = \frac{x}{L}$$

'L' being a characteristic length.

5. Characteristic parameters for $\wedge_m = 0, 1, 2$

We record below the analytical expressions for the flow parameters in the various cases, considered above.

(i) $\wedge_m = 0$

$$\frac{\tau(\prime)\sqrt{Re}}{\rho\, U_0^2} = \frac{\overline{U}(\overline{x})}{\sqrt{\dfrac{\wedge_p}{\overline{U}'}}} \; \frac{4 + \wedge_p}{3} \tag{5.1}$$

$$\overline{\delta}^*\sqrt{Re} = \sqrt{\frac{\wedge_p}{\overline{U}'}} \; \frac{12 - \wedge_p}{30} \tag{5.2}$$

$$\overline{\theta}\sqrt{Re} = \sqrt{\frac{\wedge_p}{\overline{U}'}} \; \frac{(1624 - 22\wedge_p - 17\wedge_p^2)}{11340} \tag{5.3}$$

where $\quad \delta^* = \dfrac{\delta}{L}, \; \overline{\theta} = \dfrac{\theta}{L}$

(ii) $\wedge_m = 1$

$$\frac{\tau(x)\sqrt{Re}}{\rho\, U_0^2} = \sqrt{\frac{\overline{U}(\overline{x})}{\dfrac{\wedge_p}{\overline{U}'}}} \; \frac{6(5 + \wedge_p)}{19} \tag{5.5}$$

$$\overline{\delta}\sqrt{Re} = \sqrt{\frac{\wedge_p}{\overline{U}'}} \; \frac{(+33 - 35\wedge_p}{1140} \tag{5.5a}$$

$$\overline{\theta}\, Re = \sqrt{\frac{\wedge_p}{\overline{U}'}} \; \frac{(129185 - 3002\wedge_p - 1159\wedge_p^2)}{909720} \tag{5.6}$$

(iii) $\wedge_m = 2$

$$\frac{\tau(x)\sqrt{Re}}{\rho\, U_0^2} = \sqrt{\frac{\overline{U}(\overline{x})}{\dfrac{\wedge_p}{\overline{U}'}}} \; \frac{6(6 + \wedge_p)}{20} \tag{5.7}$$

$$\overline{\delta}^*\sqrt{Re} = \sqrt{\frac{\wedge_p}{\overline{U}'}} \; \frac{(436 - 34\wedge_p)}{1200} \tag{5.8}$$

$$\overline{\theta}\sqrt{Re} = \sqrt{\frac{\wedge_p}{\overline{U}'}} \; \frac{(17788 - 524\wedge_p - 137\wedge_p^2)}{12600} \tag{5.9}$$

TABLE 1

The Characteristic boundary layer parameters for 4 P_2 Profile with $\wedge m = 0$

$\wedge p$	$\bar{x}$	$\dfrac{\tau(x)\sqrt{Re}}{\rho\,U_0^2}$	$\bar{\delta}^*\sqrt{Re}$	$\bar{\theta}\,\sqrt{Re}$
$\bar{U} = 1 - \bar{x}$				
0	0	∞	0	0
−0·5	0·0250	1·60848	0·2946	0·10162
−1·0	0 0468	0·9532	0·4333	0·14365
−2·0	0·0796	0·4332	0·6602	0·19954
−3·0	0·1005	0·1729	0·8660	0·23470
−4·0	0 1116	0	1·0666	0·2540
−5·0	0·1139	−·1320	1·2674	0· 580

TABLE 2

The Characteristic boundary layer parameters for 4 P_2 Profile with $\wedge m = 1$

$\wedge p$	$\bar{x}$	$\dfrac{\tau(x)\sqrt{Re}}{\rho\,U_0^2}$	$\bar{\delta}^*\sqrt{Re}$	$\bar{\theta}\,\sqrt{Re}$
$U = 1 - \bar{x}$				
0	0	∞	0	0
−0·5	0·0257	1·9580	0·2794	·1016
−1·0	0·0481	1·2024	0·4105	·1443
−2·0	0·0836	0·6135	0· 242	·2034
−3·0	0·1070	0·3256	0·8173	·2438
−4·0	0·1210	0·1388	1·0052	·2704
−5·0	0·1262	0	1·1929	·2840
−5·5	0·1263	0·0616	1·2864	·2860

TABLE 3

The Characteristic boundary layer parameters for 4 P_2 Profile with $\wedge m = 2$

$\wedge p$	$\bar{x}$	$\dfrac{\tau(x)\sqrt{Re}}{\rho\,U_0^2}$	$\bar{\delta}^*\sqrt{Re}$	$\bar{\theta}\,\sqrt{Re}$
$\bar{U} = 1 - \bar{x}$				
0	0	∞	0	0
−1·0	0·0499	1·4252	0·3916	0·1443
−2·0	0·0889	0·7727	0 5943	0·2052
−3·0	0·1182	0·4582	0·7764	0·2492
−4·0	0·1388	0·2584	0·9532	0·2808
−5·0	0·1524	0·1136	1·1296	0·3014
−6·0	0·1593	0	1·3065	0·31108
6·5	0·1610	−0·4937	1·3956	0·31164

6 Discussion of the results

In Fig. 1, we have drawn the graphs for the variation of the shape factor $\Lambda_p = \delta^2 U^1/\nu$ against the plate length $\bar{x}$ when $\Lambda_m = 0$, 1 and 2. For a given outer velocity distribation $\bar{U} = 1 - \bar{x}$; we can evaluate the boundary layer thickness δ at any point on the plate, which enables us to correlate the various characteristics of the flow with the plate length. Further we find that at any point on the plate the prescribed maganetic field reduces the boundary layer thickness.

Fig. 2 shows the variation of the skin-friction coefficient with the plate length as given by the Blasius type of series solution, the classical Kármán-Pohlhansen method and the present procedure in the absence of magnetic field for $\bar{U} = 1 - \bar{x}$. We find that the present procedure, series solution and the usual Pohlhausen Profile give the point of separation as $\bar{x}_s = 0 \cdot 1116$, $0 \cdot 120$, $0 \cdot 1519$ respectively. Without discussing the question of validity of the boundary layer equations near the point of separation, we record here that the present profile enables us to cross the point of separation and go a little distance downstream of the plate where we obtain profiles with back flow. The present profile ceases to hold beyond $\bar{x}_s = 0 \cdot 1139$ (Table 1). This seems to be a significant achievement in view of the fact that several exact procedures such as Görtler's new series
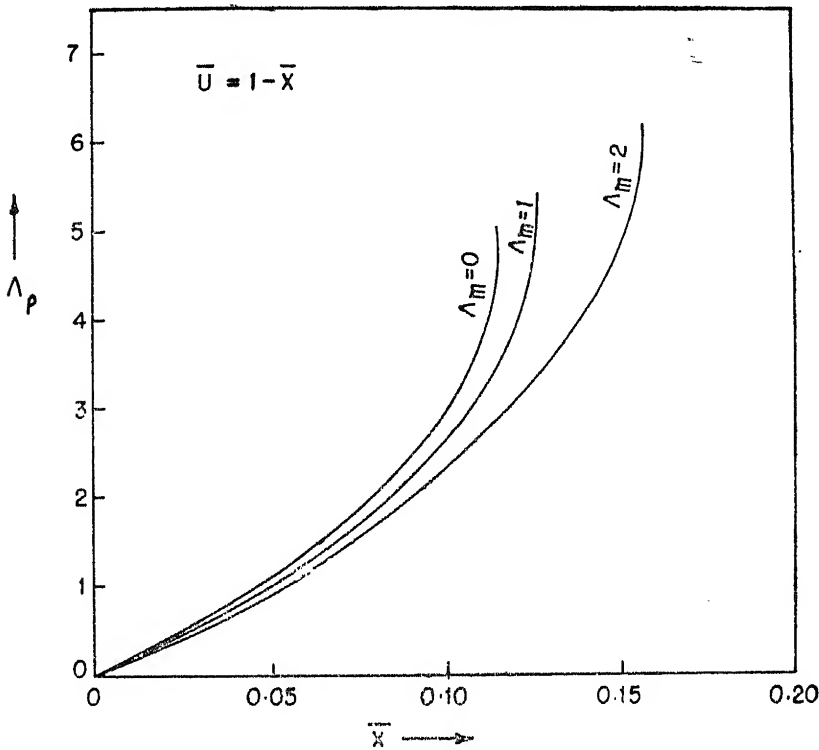
FIG. 1. VARIATION OF Λ_p WITH $\bar{X}$ FOR CONSTANT VALUES OF Λ_m
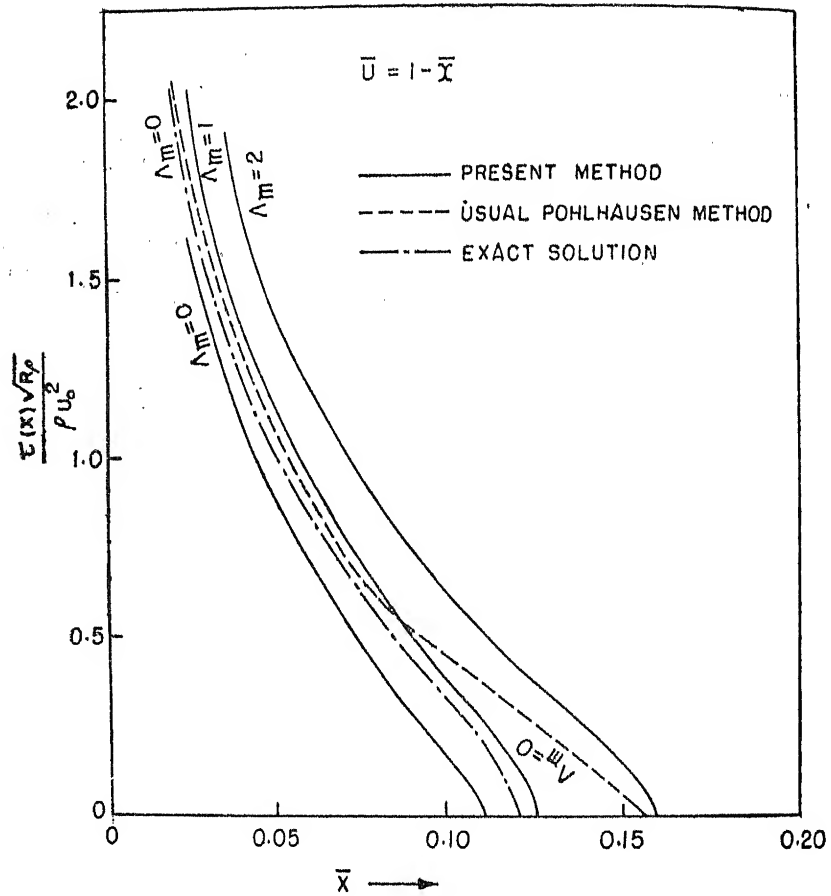
FIG. 2 VARIATION OF WALL SHEARING STRESSES
WITH MAGNETIC FIELD PARAMETER Λ_m
AGAINST THE PLATE LENGTH $\bar{X}$.
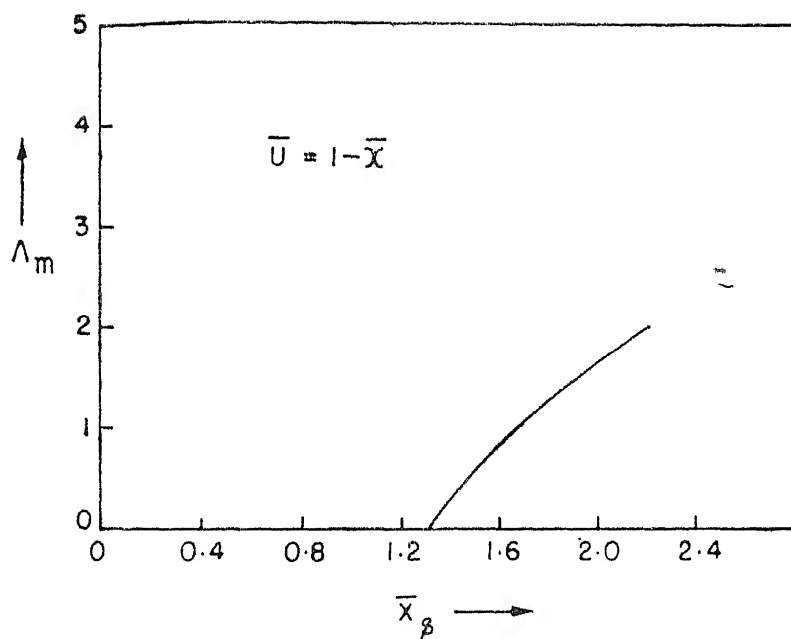
FIG. 3.   POINT OF SEPARATION $\overline{X}_\beta$ AS A
FUNCTION OF TRANSVERSE MAGNETIC
FIELD PARAMETER.
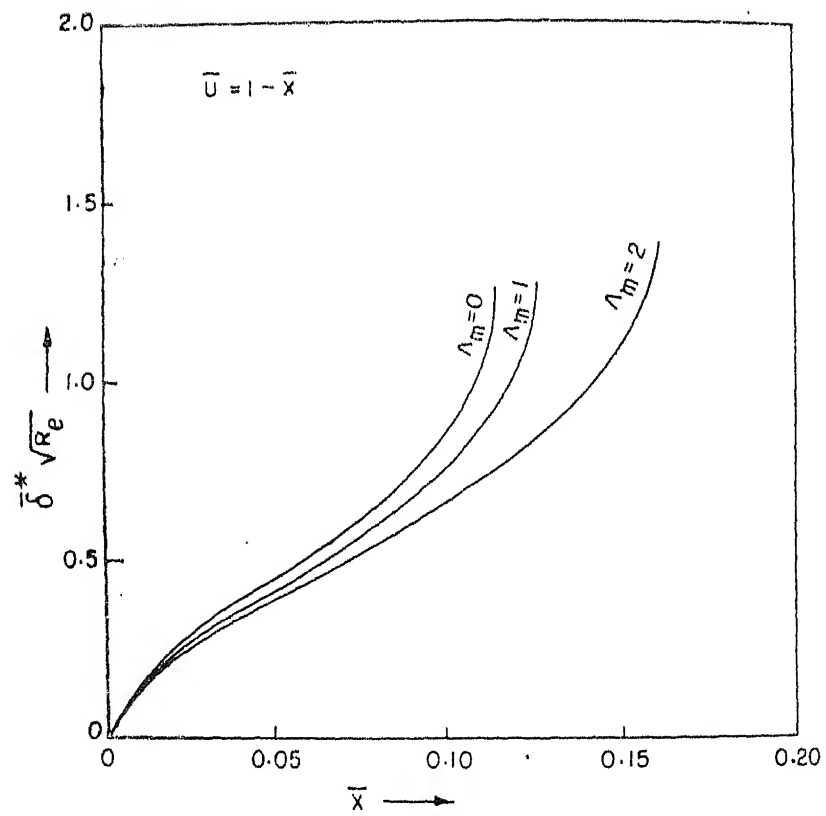
FIG. 4. VARIATION OF $\overline{\delta}^*$ WITH MAGNETIC FIELD

$$\frac{\partial}{\partial x} B_n(x^g, y^h)$$

$$= \frac{ng}{x} B_n(x^g, y^h) - mh \sum_{K=1}^{\left[\frac{n}{m}\right]} \frac{[(\alpha_p)]_{n-mk} [(a_r)]_k \, \mu^k \, \nu^{n-mk} \, x^{g(n-mk)-1}}{(k-1)! \, (n-mk)! \, [(\beta_q)]_{n-mk} [(b_s)]_k \, y^{mhk}}$$

$$= \frac{ng}{x} B_n(x^g, y^h) - mh \sum_{K=0}^{\left[\frac{n-m}{m}\right]} \frac{[(\alpha_p)]_{n-m-mk} [(a_r)]_{k+1} \, \mu^{k+1} \, \nu^{n-m-mk} \, x^{g(n-m-mk)-1}}{k! \, (n-m-mk)! \, [(\beta_q)]_{n-m-mk} [(b_s)]_{k+1} \, y^{mh(k+1)}}$$

$$= \frac{ng}{x} B_n(x^g, y^h) - \frac{mh\,\mu[(a_r)]}{xy^{mh}[(b_s)]} B_{n-m}^{(a_r)+1\,;\,(b_s)+1}(x^g, y^h), \qquad (5.10)$$

where $n \geqslant m$.

($viii$) By the method similar to that in (vii), we easily arrive at

$$\frac{\partial}{\partial y} \{ y^{nh} B_n(x^g, y^h) \} = hy^{nh-1} \left\{ n\, B_n(x^g, y^h) - \frac{m\,\mu\,[(a_r)]}{y^{mh}[(b_s)]} B_{n-m}^{(a_r)+1\,;\,(b_s)+1} (x^g, y^h) \right\},$$
$$(5.11)$$

where $n \geqslant m$.

(ix) From (5.6) and (5.10), it is easy to see that

$$\left(\frac{x}{g} \frac{\partial}{\partial x} \right) B_n(x^g, y^h) - \left(\frac{y}{h} \frac{\partial}{\partial y} \right) B_n(x^g, y^h) = n\, B_n(x^g, y^h). \qquad (5.12)$$

(x) *Pure recurrence relation* : From (5.2) and (5.10), we get

$$n\, B_n(x^g, y^h) = \frac{\nu x^g [(\alpha_p)]}{[(\beta_q)]} B_{n-1}^{\ ;\,(a_p)+1\,;\,(\beta_q)+1} (x^g, y^h) + \frac{m\,\mu[(a_r)]}{y^{mh}[(b_s)]} \times$$
$$B_{n-m}^{(a_r)+1\,;\,(b_s)+1} (x^g, y^h), \qquad (5.13)$$

where $n \geqslant m$.

The following four recurrence relations can also be derived :

(xi) $\quad \dfrac{\partial}{\partial x} \{ x^c B_n(x^g, y^h) \} = (c+ng)\, x^{c-1} B_n(x^g, y^h) - \dfrac{mg\,\mu[(a_r)]\, x^{c-1}}{y^{mh}[(b_s)]} \times$

$$B_{n-m}^{(a_r)+1\,;\,(b_s)+1} (x^g, y^h). \qquad (5.14)$$

where $n \geqslant m$.

(xii) $\qquad \dfrac{\partial}{\partial x} \{ x^c B_n(x^g, y^h) \} = cx^{c-1} B_n(x^g, y^h) +$

$$\frac{g\nu[(\alpha_p)]}{[(\beta_q)]} x^{c+g-1} B_{n-1}^{\ ;\,(a_p)+1\,;\,(\beta_q)+1} (x^g, y^h) \qquad (5.15)$$

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where $n \geqslant 1$.

$(xiii)$
$$\frac{\partial}{\partial y} \{ y^c B_n(x^g, y^h) \} = c y^{c-1} B_n(x^g, y^h) -$$

$$\frac{mh\, \mu [\,(a_r)\,]}{[\,(b_s)\,]}\, y^{c-mg-1}\, B_{n-m}^{(a_r)+1\,;\,(b_s)+1}\, (x^g, y^h), \qquad (5\cdot16)$$

where $n \geqslant m$ and

(xiv)
$$\left(x^{1-c} \frac{\partial}{\partial x} \right) x^c B_n(x^g, y^h) - \frac{g}{h} \left(y^{1-c} \frac{\partial}{\partial y} \right) y^c B_n(x^g, h) = \left(c + ng - \frac{cg}{h} \right) B_n(x^g, y^h).$$
$$(5\cdot17)$$

6. *Another set of recurrence relations* :

(i) From $(5\cdot1)$, we have

$$\frac{\partial}{\partial x} \{ x^{-ng} B_n(x^g, y^h) \} = -mg \sum_{K=1}^{\left[\frac{n}{m}\right]} \frac{(a_p)\,]_{n-mk}\,[\,(a_r)\,]_k\, \mu^k\, v^{n-mk}\, x^{-mgk-1}}{(k-1)!\,(n-mk)!\,[\,(\beta_q)\,]_{n-mk}\,[\,(b_s)\,]_k\, y^{mhk}}$$

$$= -mg \sum_{K=0}^{\left[\frac{n-m}{m}\right]} \frac{[\,(\alpha_p)\,]_{n-m-mk}\,[\,(a_r)\,]_{k+1}\, \mu^{k+1}\, v^{n-m-mk}\, x^{-mgk-mg-1}}{k!\,(n-m-mk)!\,[\,(\beta_q)\,]_{n-m-mk}\,[\,(b_s)\,]_{k+1}\, y^{mhk+mh}}$$

$$= \frac{(-mg\, \mu)\,[\,(a_r)\,]}{y^{mh}\, x^{ng+1}\,[\,(b_s)\,]}\, B_{n-m}^{(a_r)+1\,;\,(b_s)+1}\, (x^g, y^h), \qquad (6\cdot1)$$

when $n \geqslant m$.

(ii) Relation $(6\cdot1)$ can be thrown in the form

$$\left(x^{mg+1} \frac{\partial}{\partial x} \right) \{ x^{-mg} B_n(x^g, y^h) \} = \frac{(-mg\, \mu)\,[\,(a_r)\,]}{x^{g(n-m)}\, y^{mh}\, [(b_s)]}\, B_{n-m}^{(a_r)+1\,;\,(b_s)+1}\, (x^g, y^h), \quad (6\cdot2)$$

from which on operating the operator $\left(x^{mg+1} \frac{\partial}{\partial x} \right)$ k times, we obtain

$$\left(x^{mg+1} \frac{\partial}{\partial x} \right)^k x^{-mg} B_n(x^g, y^h) = \frac{(-mg\, \mu^{-k})\,[\,(a_r)\,]_k}{x^{g(n-mk)}\, y^{mhk}\, [(b_s)]_k}\, B_{n-mk}^{(a_r)+k\,;\,(b_s)+k} (x^g, y^h),$$

where $n \geqslant mk$ and k is a non–negative integer. $\qquad (6\cdot3)$

Similarly, the following two recurrence relations can also be obtained :

(iii)
$$\left(y^{1-h} \frac{\partial}{\partial y} \right) \{ y^{nh} B_n(x^g, y^h) \} = \frac{vh\, x^g\, [\,(\alpha_p)\,]}{y^{h(1-n)}\, [(\beta_q)]}\, B_{n-1}^{(\alpha_p)+1\,(\beta_q)+1} (x^g, y^h),$$
$$(6\cdot4)$$

where $n \geqslant 1$ and

(iv)
$$\left(y^{1-h} \frac{\partial}{\partial y} \right)^k \{ y^{nh} B_n(x^g, y^h) \} = \frac{(vh)^k x^{gk}\, [\,(\alpha_p)\,]_k}{y^{h(k-n)}\, [(\beta_q)]_k}\, B_{n-k}^{(\alpha_p)+k\,;\,(\beta_q)+k} (x^g, y^h),$$

where $n \geqslant k$ and k is a non–negative integer. $\qquad (6\cdot5)$

(v) Again, on using (6·4) in (6·2), we arrive at

$$\left(\frac{x^{mg+1}}{y^{h-1}} \frac{\partial^2}{\partial y\, \partial x} \right) \{ x^{-ng}\, y^{nh}\, B_n(x^g, y^h) \}$$

$$= \frac{(-mgh\,\mu\nu)\, y^{h(n-m-1)}\, [\,(a_r)\,]\, [\,(\alpha_p)\,]}{x^{g(n-m-1)}\, [\,(b_s)\,]\, [\,(\beta_q)\,]}\; B^{(a_r)+1\,;\,(b_s)+1}_{n-m-1\,;\,(\alpha_p)+1\,;\,(\beta_q)+1}\, (x^g, y^h), \qquad (6·6)$$

where $n \geqslant m+1$, and continuing this process k–times, we get

$$\left(\frac{x^{mg+1}}{y^{h-1}} \frac{\partial^2}{\partial y\, \partial x} \right)^k \{ x^{-ng}\, y^{nh}\, B_n(x^g, y^h) \}$$

$$= \frac{(-mgh\,\mu\nu)^k\, y^{h(n-mk-k)}\, [\,(a_r)\,]_k\, [\,(\alpha_p)\,]_k}{x^{g(n-mk-k)}\, [\,(b_s)\,]_k\, [\,(\beta_q)\,]_k}\; B^{(a_r)+k\,;\,(b_s)+k}_{n-mk-k\,;\,(\alpha_p)+k\,;\,(\beta_q)+k}\, (x^g, y^h), \qquad (6·7)$$

where $n \geqslant (m+1)\, k$ and k is a non-negative integer.

7. *The partial differential equation for $B_n(x, y)$:*

Let
$$W_1 = {}_{mq+m+r}F_{mp+s}\left[\begin{array}{c} \triangle(m\,;\,-n),\ \triangle(m\,;\,1-(\beta_q)-n),\ (a_r) \\ \triangle(m\,;\,1-(\alpha_p)-n),\ (b_s) \end{array} \; ;\, c\,x^{-m} \right]$$

Since $\theta\, x^{-m} = -m\, x^{-m}$, where $\theta = x\, \dfrac{\partial}{\partial x}$, it follows that

$$\{ \theta[\,(b_s) - \frac{\theta}{m} - 1] \triangle[m\,;\, 1-(\alpha_p) - n - m - \theta] \}W_1$$

$$= \sum_{K=1}^{\left[\frac{n}{m}\right]} \frac{(-mk)c^k\triangle_k[m;\,-n]\triangle_k[m;\,1-(\beta_q)-n]\, [\,(a_r)]_k[k+(b_s)-1]\triangle[m;\,1-(\alpha_p)-n+mk\ m]}{k!\,[(b_s)]_k\,\triangle_k\,[m\,;\,1-(\alpha_p)-n]\, x^{mk}}$$

$$= -m \sum_{K=1}^{\left[\frac{n}{m}\right]} \frac{\triangle_k[m\,;\,-n]\,\triangle_k[m\,;\,1-(\beta_q)-n]\,[(a_r)]_k\, c^k x^{-mk}}{(k-1)!\,[(b_s)]_{k-1}\,\triangle_{k-1}[m\,;\,1-(\alpha_p)\ n]}$$

$$= -m \sum_{K=0}^{\left[\frac{n-m}{m}\right]} \frac{\triangle_{k+1}[m\,;\,-n]\,\triangle_{k+1}[m\,;\,1-(\beta_q)-n]\,[(a_r)]_{k+1}\, c^{k+1}\, x^{-m(k+1)}}{k!\,[(b_s)]_k\,\triangle_k[m\,;\,1-(\alpha_p)-n]}.$$

Since $\triangle[m\,;\,-n+km]$, the last m factors in the product $\triangle_{k+1}[m\,;\,-n]$, is zero in the interval $\dfrac{n-m+1}{m} \leqslant k \leqslant \dfrac{n}{m}$, we consequently have

$$\{ \theta\,[(b_s) - \frac{\theta}{m} - 1]\, \triangle[m\,;\, 1-(\alpha_p)-n-m-\theta] \}\, W_1$$

$$= -mc \sum_{K=0}^{\left[\frac{n}{m}\right]} \frac{\triangle_k [m \ ; \ -n] \ \triangle_k [m \ ; \ 1-(\beta_q)\cdot n] \ [(a_r)]_k \ c^k \ x^{-mk}}{k \ ! \ (b_s)\_k \ \triangle_k[m \ ; \ 1-(\alpha_p)-n]} \times$$

$$\times \ \{\triangle[m \ ; \ -n+km] \ \triangle[m \ ; \ 1-(\beta_q)-n+mk] \ [(a_r)+k] \ x^{-m}\}.$$

i.e. $\quad \{\upsilon[(b_s) - \frac{\theta}{m} - 1] \ \triangle[m \ ; \ 1-(\alpha_p)-n-m \ \theta] + cmx^{-m} \ \triangle[m \ ; \ -n-\theta] \ [(a_r) - \frac{\theta}{m}] \times$

$$\triangle[m \ ; \ 1-(\beta_q)-n\cdot\theta]\} \ W_1 = 0.$$

i.e. $\{\theta \ [\theta-m(b_s)+m] \ [\theta-(m)+(\alpha_p)+n+m] -c(-m)^{m(p-q-1)+s-r+1}$

$$\times \ [\theta+n+1-(m)] \ [\theta-(m)+(\beta_q)+n] \ [\theta-m(a_r)]\} \ W_1 = 0. \tag{7.1}$$

Again, let $W = Hx^n W_1 \therefore W_1 = \frac{1}{H} x^{-n} W$

Hence $\theta \{W_1\} = \theta\left\{\frac{1}{H} x^{-n} W\right\} = \frac{1}{H} \{x^{-n} \theta \ W - nx^{-n}W\} = \frac{1}{H} x^{-n} \ (\theta-n) \ W.$

Therefore, from (7.1), we get

$$\frac{1}{H} \ x^{-n} \ \{ \ (\theta-n) \ [\theta-n-m \ (b_s)+m] \ [\theta-(m)+(\alpha_p)+m] -c(-m)^{m(p-q-1)+s-r+1}$$

$$\times \ [\theta-(m)+1] \ [\theta-(m)+(\beta_q)] \ [\theta-n-m(a_r)] \ \} \ W = 0.$$

Now, when $C = \frac{(-m)^{m(q-p+1)}\mu}{(\nu y)^m}$ and $H = \frac{[(\alpha_p)]_n \ (\nu x)^n}{n \ ! \ [(\beta_q)]_n}$

then $W = B_n(x, y)$,

$$\therefore \{ \ (\theta-n) \ [\theta-n-m \ (b_s)+m] \ [\theta-(m)+(\alpha_p)+m]- \frac{(-m)^{s-r+1} \ \mu}{(\nu \ xy)^m} \ [\theta-(m)+1]$$

$$\times \ [\theta-(m)+(\beta_q)] \ [\theta-n-m \ (a_r)] \} \ B_n(x, y) = 0. \tag{7.2}$$

Similarly, we can get the differential equation in ψ where $\psi = y \frac{\partial}{\partial y}$; in the form

$$\{\psi \ [\psi-m(b_s)+m] \ [\psi-(m)+(\alpha_p)+n+m] - \frac{(-m)^{s-r+1}}{(\nu \ xy)^m} \ \mu \ [\psi+n-(m)+1]$$

$$\times \ [\psi+(\beta_q)+n-(m)] \ [\psi-m(a_r)] \} \ B_n(x, y) = 0. \tag{7.3}$$

Hence, the final form of the partial differential equation can be written as follows :

$$\{ \ (\theta-n) \ [\theta-n-m \ (b_s)+m] \ [\theta-(m)+(\alpha_p) \ +m] \ +\psi \ [\psi-m(b_s)+m]$$

$$\times \ [\psi-(m)+(\alpha_p)+n+m] - \frac{(-m)^{s-r+1} \mu}{(\nu \ xy)^m} \ \{ \ [\theta-(m)+1] \ [\theta-(m)+(\beta_q) \ ]$$

$$\times [\theta-n-m(a_r)] + [\psi+n-(m)+1] \ [\psi+(\beta_q)+n-(m)] \ [\psi-m(a_r)]\} \} \ B_n(x, y) = 0. \tag{7.4}$$

Amino acid Content of the Trunk Bark of Pterospermum acerifolium

By

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Pterospermum acerifolium[1,2] (Hindi—Kanak Champa, N. O.—Leguminosae) which grows in sub-Himalayan tract, outer Himalayan valleys and hills (upto a height of 400 feet), Bengal, Chittagong, Khasia hills and Burma, has been investigated with a view to determine the amino acid content of its trunk bark.

In view of the great medicinal importance of this plant, it was thought desirable to carry out systematic chemical investigation of the trunk bark. In our earlier communications we have reported the chemical examination of the oil from the seed,[3] the fat from the trunk bark[4] and the seeds.[5] In the present communication we are giving the amino acid contents of the trunk bark. Water, 0·2% NaOH and 80% alcohol were employed to isolate the protein from the trunk bark. The extracted protein matter was hydrolysed with HCl (6N). The hydrolysates were examined by the descending paper chromatographic technique for the characterization of different amino acids present in them. The protein of the trunk bark of Pterospermum acerifolium was found to be made up of glycine, tyrosine, cystine, alanine and leucine. The trunk bark of Pterospermum acerifolium contains the same amino acids as the seed except that one more amino acid, leucine, is present in the former.

Experimental

Isolation of Protein Matter from Trunk Bark :

The trunk bark was dried, powdered and soxhleted with pertoleum ether (b.p. 60–80°). The defatted trunk bark was treated successively with water, 0·2% NaOH and 80% ethanol.

Finely powdered material (500 g.) was mixed with 1 litre of distilled warm water, stirred and then filtered. The residue (No. 1) was shaken with 800 ml. of 0·2% NaOH solution for three hours and filtered. The residue (No. 2) was washed with water to free it from NaOH and then refluxed with 80% ethanol for 24 hours and filtered. The residue (No. 3) did not give a positive test of nitrogen indicating thereby that no more protein was present in the trunk bark.

Hydrolysis of the Protein :

100 ml. of each of the extracts, was refluxed with 25 ml. HCl (6N) for 16–20 hours on a water bath. Each of the hydrolysates was then taken in a porcelain dish separately and the acid was removed by evaporation over a water bath.

Evaporation was continued repeatedly by adding distilled water from time to time, till the vapours did not give white fumes with ammonia. Each of the hydrolysates was then extracted with absolute alcohol, and all the three extracts were mixed.

Characterization of Amino Acids :

Amino acids were characterized by employing descending paper chromato-graphic technique. The mixed ethanolic extract of the protein hydrolysates was chromatographed together with reference amino acids on Whatman No. 1 filter paper. The chromatogram was equilibrated with the lower layer on *n*-butanol: acetic acid : water (4 : 1 : 5 v/v) for 20 hours and was developed by the upper layer of the above mixture for 20 hours at room temperature. After drying the developed chromatogram in air for 24 hours, it was sprayed with $0 \cdot 1\%$ solution of ninhydrin in acetone. Finally, it was kept in an electric oven at 55°C for 5-10 minutes and the amino acids were located as spots.

The Rf values of the reference acids, corresponding to the spots of the amino acids from protein hydrolysates were calculated and are recorded in Table 1. The amino acids present in the protein hydrolysates are recorded in Table 2.

TABLE 1	TABLE 2
Rf Value of Amino Acids at 30°	Amino Acids Present in the Protein Hydrolystates

Rf Value	Amino acid	
0·05	Cystine	Cystine, Glycine,
0·08	Glycine	Alanine, Tyrosine,
0·26	Alanine	and Leucine.
0·33	Tyrosine	
0·70	Leucine	

Thus, the amino acids, present in trunk bark of Pterospermum acerifolium are cystine, glycine, alanine, tyrosine and leucine.

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On some Finite integrals involving Generalized G--function

By

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Abstract

In this paper first we evaluate two integrals involving Generalized G-function in two variables introduced by Prof. Agarwal [(1), p. 537]. Later on we find sum of a finite series of these functions. Then we establish an interesting expansion formula for this function. Various integrals found recently by different authors follow as special cases of our results.

1 Introduction

Agarwal [(1), p. 537] gave a generalization of Meijer's G-function in two variables by means of double Mellin–Barne's contour integral in the form*

$$
G \begin{array}{l} p,\, q,\, s,\, r,\, t' \\[4pt] A,\, [C,\, E],\, B,\, [D,\, F] \end{array} \left[y \left| \begin{array}{l} (a) \\ (c);\, (e) \\ (b) \\ (d);\, (f) \end{array} \right. \right]
$$

$$
= \left(\frac{1}{2\pi i} \right)^2 \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \Phi(\xi+\eta)\, \psi(\xi,\, \eta)\, x^{\xi}\, y^{\eta}\, d\xi\, d\eta \qquad (1\cdot 1)
$$

where

$$
\Phi(\xi+\eta) = \frac{\prod\limits_{j=1}^{p} \Gamma(1 - a_j + \xi + \eta)}{\prod\limits_{j=p+1}^{A} \Gamma(a_j - \xi - \eta)\; \prod\limits_{j=1}^{B} \Gamma(b_j + \xi + \eta)},
$$

$$
\psi(\xi,\, \eta) = \frac{\prod\limits_{j=1}^{q} \Gamma(c_j + \xi)\; \prod\limits_{j=1}^{r} \Gamma(d_j - \xi)\; \prod\limits_{j=1}^{s} \Gamma(e_j + \eta)\; \prod\limits_{j=1}^{t'} \Gamma(f_j - \eta)}{\prod\limits_{j=q+1}^{C} \Gamma(1 - c_j - \xi)\; \prod\limits_{j=r+1}^{D} \Gamma(1 - d_j + \xi)\; \prod\limits_{j=s+1}^{E} \Gamma(1 - e_j - \eta)\; \prod\limits_{j=t'+1}^{F} \Gamma(1 - f_j + \eta)},
$$

$$
0 \leqslant p \leqslant A,\;\; 0 \leqslant q \leqslant C,\;\; 0 \leqslant r \leqslant D,\;\; 0 \leqslant s \leqslant E,\;\; 0 \leqslant t' \leqslant F.
$$

*Though the notation for the G-function in two arguments that we discuss here differs from the one given earlier by Prof. Agarwal, in essence the function remains same.

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Here as well as in what follows the symbol (a) is used to represent sequence of A parameters $a_1, \ldots, a_A$, $(a_{m,p})$ will denote sequence of $(p-m)$ parameters $a_{m+1}, \ldots, a_p$ and (a_m) will denote sequence of m-parameters $a_1, \ldots, a_m$.

The sequence of parameters (a_p), (c_q), (d_r), (e_s) and $(f_{t'})$ are such that none of the poles of the integrand coincide, the paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(d_j - \xi)$, $(j = 1, \ldots, r)$ and $\Gamma(f_k - \eta)$, $(k = 1, \ldots, t')$ lie to the right and those of $\Gamma(c_j + \xi)$, $(j = 1, \ldots, q)$, $\Gamma(e_k + \eta)$, $(k = 1, \ldots, s)$ and $\Gamma(1 - a_j + \xi + \eta)$, $(j = 1, \ldots, p)$ lie to the left of the imaginary axis.

We will represent left hand side of $(1\cdot1)$ by $G\begin{bmatrix} x \\ y \end{bmatrix}$.

The integral $(1\cdot1)$ converges, under the following conditions :

$$
\left.
\begin{aligned}
& 2(p+q+r) > A+B+C+D, \quad 2(p+s+t') > A+B+E+F \\
& |\arg x| < \left(p + q + r - \frac{A}{2} - \frac{B}{2} - \frac{C}{2} - \frac{D}{2} \right) \pi \\
& |\arg y| < \left(p + s + t' - \frac{A}{2} - \frac{B}{2} - \frac{E}{2} - \frac{F}{2} \right) \pi
\end{aligned}
\right\}
\qquad (1\cdot2)
$$

The behaviour of $G\begin{bmatrix} x \\ y \end{bmatrix}$ for small x and y is given by

$$
G\begin{bmatrix} x \\ y \end{bmatrix} = 0 \left(|x|^{\alpha} |y|^{\beta} \right),
$$

where $\quad \alpha = \min. R(d_j), \quad (j = 1, \ldots, r)$

$\qquad \beta = \min. R(f_k), \quad (k = 1, \ldots, t')$ $\qquad\qquad (1\cdot3)$

The equation $(1\cdot1)$ reduces to product of two G-functions if we take $A = B = 0$, and we shall have then

$$
G \begin{matrix} 0, q, s, r, t' \\ 0, [C, E], 0, [D, F] \end{matrix} \begin{bmatrix} x \\ y \end{bmatrix} \begin{matrix} (c) ; (e) \\ (d) ; (f) \end{matrix}
$$

$$
= G^{r, q}_{C, D}\begin{bmatrix} x & \Big| & 1 - (c) \\ & & (d) \end{bmatrix} \; G^{t', s}_{E, F}\begin{bmatrix} y & \Big| & 1 - (e) \\ & & (f) \end{bmatrix} \qquad (1\cdot4)
$$

and if we take $p = A$, $s = E$, $t' = 1$, $f_1 = 0$, and taking $y \to 0$, the generalized G-function reduces to Meijer's G-function, we have

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$$\underset{y\to 0}{Lt}\; G^{A,\,q,\,E,\,r,\,1}_{A,\,[C,\,E],\,B,\,[D,\,F]}\left[\begin{matrix} x \\ y \end{matrix}\;\middle|\;\begin{matrix} (c)\overset{(a)}{;}(e) \\ (d)\;;\;0,(f_1,{}_F) \end{matrix}\right]$$

$$=\frac{\overset{E}{\underset{j=1}{\text{II}}}\,\Gamma(e_j)}{\overset{F}{\underset{j=2}{\text{II}}}\,\Gamma(1-f_j)}\; G^{r,\,A+q}_{A+C,\,B+D}\left[x\;\middle|\;\begin{matrix}(a,\;1-(c) \\ (d),\;1-(b)\end{matrix}\right] \tag{1.5}$$

Also the following symbols will be used throughout the paper ;

$$(\alpha)_r = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \tag{1.6}$$

$$\triangle(n,a)=\frac{a}{n},\;\frac{a+1}{n},\;\ldots,\;\frac{a+n-1}{n} \tag{1.7}$$

and $_pF_q$ will denote the generalized hypergeometric function [(7), p. 73].

During the course of the proof we shall also make use of the following results

$$E_\alpha\,f(\alpha)=f(\alpha+1) \tag{1.8}$$

$$E_\alpha{}^\nu f(\alpha)=f(\alpha+\nu) \tag{1.9}$$

2. *Finite integrals :*

First Integral :

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\, G\left[\begin{matrix} yx^\lambda \\ zx^\lambda \end{matrix}\right]dx = \lambda^{-\sigma}\,t^{\rho+\sigma-1}\,\Gamma(\sigma)\;\times$$

$$\times\; G^{p+\lambda,\,q,\,s,\,r,\,t'}_{A+\lambda,\,[C,\,E],\,B+\lambda,\,[D,\,F)}\left[\begin{matrix} yt^\lambda \\ zt^\lambda \end{matrix}\;\middle|\;\begin{matrix} \triangle(\lambda,1-\rho),\;(a) \\ (c)\;;\;(e) \\ (b),\triangle(\lambda,\rho+\sigma) \\ (d)\;;\;(f) \end{matrix}\right] \tag{2.1}$$

The above formula is valid under the following conditions :

$R(\rho)>0,\,R(\sigma)>0,\,\lambda>0,\,R(\rho+\lambda(d_j+f_k))>0,\,(j=1,\ldots,r;k=1,\ldots t')$

and $2(p+q+r)>(A+B+C+D),\,2(p+s+t')>(A+B+E+F)$,

$$\mid\arg y\mid<\left(p+q+r-\frac{A}{2}-\frac{B}{2}-\frac{C}{2}-\frac{D}{2}\right)\pi,$$

$$\mid\arg z\mid<\left(p+s+t'-\frac{A}{2}-\frac{B}{2}-\frac{E}{2}-\frac{F}{2}\right)\pi.$$

Second Integral :

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} \; {}_uF_v \; (\alpha_1, \ldots, \alpha_u \; ; \beta_1, \ldots, \beta_v; \; kx^\nu \, (t-x)^\mu) \; G\begin{bmatrix} yx^\lambda \\ zx^\lambda \end{bmatrix} dx$$

$$= \lambda^{-\sigma} t^{\rho+\sigma-1} \sum_{m=0}^{\infty} \frac{\prod\limits_{j=1}^{u} (\alpha_j)_m \; k^m \; t^{m(\mu+\nu)} \; \Gamma(\sigma+\mu m)}{\prod\limits_{j=1}^{v} (\beta_j)_m \; m! \; \lambda^{\mu m}} \times$$

$$\times G\begin{bmatrix} p+\lambda, q, s, r, t' \\ A+\lambda, [C,E], B+\lambda, [D,F] \end{bmatrix}\begin{bmatrix} yt^\lambda \\ zt^\lambda \end{bmatrix}\begin{vmatrix} \triangle(\lambda, 1-\rho-\nu m), (a) \\ (c \quad ; (e) \\ (b), \triangle(\lambda, \rho+\sigma+m(\nu+\mu)) \\ (d) \; ; (f) \end{vmatrix} \quad (2 \cdot 2)$$

where ν, μ are positive integers (either ν or μ may be zero). The formula $(2 \cdot 2)$ holds if $u \leqslant v+1$

no one of $\beta_1, \ldots, \beta_v$ is zero or a negative integer, $R(\sigma) > 0$, $R(\rho) > 0$, $\lambda > 0$, $2(p+q+r) > (A+B+C+D)$, $2(p+s+t') > (A+B+E+F)$,

$$|\arg y| < \left(p+q+r-\frac{A}{2}-\frac{B}{2}-\frac{C}{2}-\frac{D}{2}\right)\pi,$$

$$|\arg z| < \left(p+s+t'-\frac{A}{2}-\frac{B}{2}-\frac{E}{2}-\frac{F}{2}\right)\pi \text{ and}$$

$R(\rho + \lambda (d_j + f_k)) > 0, \; (\, j = 1, \ldots, r \; ; k = 1, \ldots, t').$

Proof of $(2 \cdot 1)$: Expressing $G\begin{bmatrix} yx^\lambda \\ zx^\lambda \end{bmatrix}$ in terms of double contour integral by $(1 \cdot 1)$ and then changing the order of integration, we have

$$\left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \Phi(\xi+\eta) \; \psi(\xi,\eta) \; y^\xi z^\eta \left\{\int_0^t x^{\rho-1+\lambda(\xi+\eta)} \, (t-x)^{\sigma-1} \; dx\right\} d\xi \, d\eta$$
$$(2 \cdot 5)$$

Evaluating the inner integral in $(2 \cdot 5)$ with the help of $[(5) \text{ p. } 10]$, we get

$$\Gamma(\sigma)\left(\frac{1}{2\pi i}\right)^2 \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \Phi(\xi+\eta) \; \psi(\xi,\eta) \; y^\xi z^\eta \; t^{\rho+\sigma-1+y(\xi+\eta)} \; \frac{\Gamma(\rho+\lambda(\xi+\eta))}{\Gamma(\rho+\sigma+\lambda(\xi+\eta))} d\xi d\eta$$

Now making use of Gauss's multiplication formula, $[(5), \text{ p. } 4]$ and interpreting the result thus obtained with the help of $(1 \cdot 1)$ we get the required result.

Regarding the interchange of the order of integration it is observed that x-integral is convergent if $R(\rho) > 0$, $R(\sigma) > 0$, $\lambda > 0$, $R(\rho + \lambda(d_j + f_k)) > 0$, $(j = 1, 2, \ldots, r, ; k = 1, 2, \ldots, t')$. The double contour integral converges if the conditions given in $(1 \cdot 2)$ are satisfied, and the convergence of repeated

integral follows from that of the integral in (2·1). Hence the interchange of the order of integration is justified by Bromwich [(2), p. 504].

Proof of (2·2) :—On multiplying both the sides of (2·1) by

$$\frac{\prod\limits_{j=1}^{u} \Gamma(\alpha_j + \varepsilon)\, k^{\varepsilon}}{\prod\limits_{j=1}^{v} \Gamma(\beta_j + \varepsilon)}\quad\text{and operating exp. } (E_\rho^{\nu}\ E_\sigma^{\mu}\ E_\varepsilon),\ \text{we get}$$

$$\text{exp. } (E_\rho^{\nu}\ E_\sigma^{\mu}\ E_\varepsilon)\int_0^t x^{\rho-1}\,(t-x)^{\sigma-1}\, G\begin{bmatrix} yx^\lambda \\ zx^\lambda \end{bmatrix}\frac{\prod\limits_{j=1}^{u}\Gamma(\alpha_j+\varepsilon)\,k^\varepsilon}{\prod\limits_{j=1}^{v}\Gamma(\beta_j+\varepsilon)}dx$$

$$= \text{exp } (E_\rho^{\nu}\ E_\sigma^{\mu}\ E_\varepsilon)\left\{\lambda^{-\sigma}\,t^{\rho+\sigma-1}\,\Gamma(\sigma)\,\frac{\prod\limits_{j=1}^{u}\Gamma(\alpha_j+\varepsilon)\,k^\varepsilon}{\prod\limits_{j=1}^{v}\Gamma(\beta_j+\varepsilon)}\times\right.$$

$$\left.\times\ G\begin{matrix} p+\lambda,\, q,\, s,\, r,\, t' \\ A+\lambda,\, [C,E],\, B+\lambda,\, [D,F] \end{matrix}\begin{bmatrix} yt^\lambda \\ zt^\lambda \end{bmatrix}\begin{vmatrix} \triangle(\lambda,\,1-\rho),\,(a) \\ (c)\,;\,(e) \\ (b),\,\triangle(\lambda,\,\rho+\sigma) \\ (d)\,;\,(f) \end{vmatrix}\right\}\qquad(2·6)$$

(2·6) reduces to the following result by virtue of (1·9)

$$\sum_{m=0}^{\infty}\int_0^t x^{\rho-1}\,(t-x)^{\sigma-1}\,G\begin{bmatrix} yx^\lambda \\ zt^\lambda \end{bmatrix}\frac{\prod\limits_{j=1}^{u}\Gamma(\alpha_j+\varepsilon+m)\,k^{\varepsilon+m}\,x^{\nu m}\,(t-x)^{\mu m}}{\prod\limits_{j=1}^{v}\Gamma(\beta_j+\varepsilon+m)\quad m\,!}$$

$$=\sum_{m=0}^{\infty}\frac{\prod\limits_{j=1}^{u}\Gamma(\alpha_j+\varepsilon+m)\,k^{\varepsilon+m}}{\prod\limits_{j=1}^{v}\Gamma(\beta_j+\varepsilon+m)\,m\,!}\ \lambda^{-\sigma-\mu m}\,t^{\rho+\sigma-1+m\,(\nu+\mu)}\,\Gamma(\sigma+\mu m)\times$$

$$\times\ G\begin{matrix} p+\lambda,\, q,\, s,\, r,\, t' \\ A+\lambda,\, [C,E],\, B+\lambda,\, [D,F] \end{matrix}\begin{bmatrix} yt^\lambda \\ zt^\lambda \end{bmatrix}\begin{vmatrix} \triangle(\lambda,\,1-\rho-\nu\,m),\,(a) \\ (c)\,;\,(e) \\ (b),\,\triangle(\lambda,\,\rho+\sigma+m\,(\mu+\nu)) \\ (d)\,;\,(f) \end{vmatrix}\qquad(2·7)$$

Using (1·7) and replacing $\alpha_j + \varepsilon$ by a_j and $\beta_j + \varepsilon$ by β_j we get the required result.

To justify the change of order of integration and summation in (2·7), we represent

$$_uF_v\left(\alpha_1, \ldots, \alpha_u ; \beta_1, \ldots, \beta_v ; kx^\nu (t-x)^\mu\right) \text{ by } \sum_{r=0}^{\infty} u_r(x)$$

where

$$u_r(x) = \frac{\prod\limits_{j=1}^{u} (\alpha_j)_r \, k^r \, x^{\nu r} \, (t-x)^{\mu r}}{\prod\limits_{j=1}^{v} (\beta_j)^r \; r!}$$

and

$$g(x) \text{ by } x^{\rho-1} (t-x)^{\sigma-1} G\left[\begin{array}{c} yx^\lambda \\ zx^\lambda \end{array}\right]$$

The following observations are now made :

(i) $u_0(x), u_1(x), \ldots$ are continuous functions in $(0, t)$,

(ii) The series $\sum\limits_{r=0}^{\infty} u_r(x)$ converge uniformly in $(0, t)$ if $u \leqslant v + 1$ and no one of $\beta_1, \ldots, \beta_v$ is zero or negative integer,

(iii) $g(x)$ has finite number of infinite discontinuties in $(0, t)$,

and (iv) $\int_0^t g(x) \, dx$ is absolutely convergent under the conditions given with (2·1).

Under the observations mentioned above, we see that the inversion of order of integration and summation in (2·7) is justified by [(3), p. 172].

3. *Particular Cases* : In (2·1) and (2·2), on taking $A = B = 0$, we get

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} \; G_{C, D}^{r, q}\left[yx^\lambda \;\middle|\; \begin{array}{c} 1-(c) \\ (d) \end{array}\right] G_{E, F}^{t', s}\left[zx^\lambda \;\middle|\; \begin{array}{c} 1-(e) \\ (f) \end{array}\right] dx$$

$$= \lambda^{-\sigma} t^{\rho+\sigma-1} \Gamma(\sigma) \; G_{\lambda, [C, E], \lambda, [D, F]}^{\lambda, q, s, r, t'}\left[\begin{array}{c} yt^\lambda \\ zt^\lambda \end{array} \;\middle|\; \begin{array}{c} \triangle(\lambda, 1-\rho) \\ (c) ; (e) \\ \triangle(\lambda, \rho+\sigma) \\ (d) ; (f) \end{array}\right] \tag{3·1}$$

valid under the conditions easily obtainable from (2·1).

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} \, {}_uF_v\left(\alpha_1, \ldots, \alpha_u ; \beta_1, \ldots, \beta_v ; kx^\nu (t-x)^\mu\right) \times$$

$$\times G_{C, D}^{r, q}\left[yx^\lambda \;\middle|\; \begin{array}{c} 1-(c) \\ (d) \end{array}\right] G_{E, F}^{t', s}\left[zx^\lambda \;\middle|\; \begin{array}{c} 1-(e) \\ (f) \end{array}\right] dx$$

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$$= \lambda^{-\sigma} t^{\rho+\sigma-1} \sum_{m=0}^{\infty} \frac{\prod\limits_{j=1}^{u} (\alpha_j)_m \; k^m \; t^{m(\nu+\mu)} \; \Gamma(\sigma+\mu m)}{\prod\limits_{j=1}^{v} (\beta_j)_m \; m! \; \lambda^{\mu m}} \times$$

$$\times \; G^{\lambda, \; q, \; s, \; r, \; t'}_{\lambda, \; [G, \, E], \; \lambda, \; [D, \, F]} \left[\begin{matrix} yt^\lambda \\ zt^\lambda \end{matrix} \; \middle| \; \begin{matrix} \triangle(\lambda, \, 1-\rho-\nu m) \\ (c) \; ; \; (e) \\ \triangle(\lambda, \, \rho+\sigma+m\,(\mu+\nu)) \\ (d) \; ; \; (f) \end{matrix} \right] \qquad (3.2)$$

The conditions under which (3.2) is valid are directly obtainable from (2.2)

(ii) If we take $p = A$, $s = E$, $t' = 1$, $f_1 = 0$ and make $z \to 0$ in (2.1) and (2.2), and then on replacing $A + C$ by A, $B + D$ by B, $A + q$ by s along with the necessary changes in the parameters, we get the following integral involving Meijer's G-function :

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} G^{r, \, s}_{A, \, B} \left[yx^\lambda \; \middle| \; \begin{matrix} (a) \\ (b) \end{matrix} \right] dx = \lambda^{-\sigma} t^{\rho+\sigma-1} \Gamma(\sigma) \times$$

$$\times \; G^{r, \; s+\lambda}_{A+\lambda, \; B+\lambda} \left[yt^\lambda \; \middle| \; \begin{matrix} \triangle(\lambda, \, 1-\rho), \, (a) \\ (b), \, \triangle(\lambda, \, 1-\rho-\sigma) \end{matrix} \right] \qquad (3.3)$$

Next if we put $t = \lambda = 1$ in (3.3), we get a known result [(6), p. 417].

$$\int_0^t x^{\rho-1} (t-x)^{\sigma-1} {}_uF_v(\alpha_1, \ldots, \alpha_u; \beta_1, \ldots, \beta_v; kx^\nu (t-x)^\mu) \times$$

$$G^{r, \, s}_{A, \, B} \left[yx^\lambda \; \middle| \; \begin{matrix} (a) \\ (b) \end{matrix} \right] dx = \lambda^{-\sigma} t^{\rho+\sigma-1} \times$$

$$\times \sum_{m=0}^{\infty} \frac{\prod\limits_{j=1}^{u} (\alpha_j)_m \; k^m \; t^{m(\nu+\mu)} \; \Gamma(\sigma+\mu m)}{\prod\limits_{j=1}^{v} (\beta_j)_m \; \lambda^{\mu m} \; m!} \times$$

$$G^{r, \; s+\lambda}_{A+\lambda, \; B+\lambda} \left[yt^\lambda \; \middle| \; \begin{matrix} \triangle(\lambda, \, 1-\rho-\nu m), \, (a) \\ (b), \, \triangle(\lambda, \, 1-\rho-\sigma-m\,(\mu+\nu)) \end{matrix} \right] \qquad (3.4)$$

If we take $h = 1$ in the result given by Chaabra and Singh [4], we see that both (3.4) and the result thus obtained are same.

(iii) If we set $u = 2$, $v = 1$, $a_1 = -n$, $a_2 = n + \alpha + \beta + 1$, $\beta_1 = \alpha + 1$, $\mu = 0$, $\nu = k = t = 1$, $\sigma = \beta + 1$, and on using the result [(7), p 254].

$${}_2F_1(-n, n+\alpha+\beta+1; \alpha+1; x) = \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\beta)}(1-2x) \text{ in (2.2) we get:}$$

$$\int_0^1 x^{\rho-1} (1-x)^\beta \, P_n^{(\alpha-\beta)} (1-2x) \; G \left[\begin{matrix} yx^\lambda \\ zx^\lambda \end{matrix} \right] dx = \frac{(1+\alpha)_n \, \Gamma(\beta+1)}{\lambda \beta^{+1}} \times$$

$$\sum_{m=0}^{n} \frac{(-1)^m \, (n+\alpha+\beta+1)_m}{(\alpha+1)_m \, m! \, (n-m)!} \; G^{p+\lambda,\, q,\, s,\, r,\, t'}_{A+\lambda,\, [C,E],\, B+\lambda,\, [D,F]} \left[\begin{matrix} y \\ z \end{matrix} \; \Bigg| \; \begin{matrix} \triangle(\lambda, 1-\rho-m),\, (a) \\ (c)\,;\,(e) \\ (b),\, \triangle(\lambda, \rho+\beta+m+1) \\ (d)\,;\,(f) \end{matrix} \right]$$

(3·5)

Also by virtue of a known result [(8), p. 981], we have

$$\int_0^1 x^{\rho-1} (1-x)^\beta \, P_n^{(\alpha,\beta)} (1-2x) \; G \left[\begin{matrix} yx^\lambda \\ zx^\lambda \end{matrix} \right] dx = \frac{(-1)^n \, \Gamma(\beta+n+1)}{n! \; \lambda \beta^{+1}} \times$$

$$\times \, G^{p+2\lambda,\, q,\, s,\, r,\, t'}_{A+2\lambda,\, [C,E],\, B+2\lambda,\, [D,F]} \left[\begin{matrix} y \\ z \end{matrix} \; \Bigg| \; \begin{matrix} \triangle(\lambda, 1-\rho+\alpha),\, \triangle(\lambda, 1-\rho),\, (a) \\ (c)\,;\,(e) \\ (b),\, \triangle(\lambda, \rho-\alpha-n),\, \triangle(\lambda, \rho+\beta+n+1) \\ (d)\,;\,(f) \end{matrix} \right]$$

(3·6)

With the help of (3·5) and (3·6) we get the following interesting result :

$$\frac{(1+\alpha)_n \, \Gamma(\beta+1) \, n! \, (-1)^n}{\Gamma(\beta+n+1)} \sum_{m=0}^{n} \frac{(-1)^m \, (n+\alpha+\beta+1)_m}{(\alpha+1)_m \, m! \, (n-m)!} \times$$

$$\times \, G^{p+\lambda,\, q,\, s,\, r,\, t}_{A+\lambda,\, [C,E],\, B+\lambda,\, [D,F]} \left[\begin{matrix} y \\ z \end{matrix} \; \Bigg| \; \begin{matrix} \triangle(\lambda, 1-\rho-m),\, (a) \\ (c)\,;\,(e) \\ (b),\, \triangle(\lambda, \rho+\beta+m+1) \\ (d)\,;\,(f) \end{matrix} \right] =$$

$$G^{p+2\lambda,\, q,\, s,\, r,\, t}_{A+2\lambda,\, [C,E],\, B+2\lambda,\, [D,F]} \left[\begin{matrix} y \\ z \end{matrix} \; \Bigg| \; \begin{matrix} \triangle(\lambda, 1-\rho+\alpha),\, \triangle(\lambda, 1-\rho),\, (a) \\ (c)\,;\,(e) \\ (b),\, \triangle(\lambda, \rho-\alpha-n),\, \triangle(\lambda, \rho+\beta+n+1) \\ (d)\,;\,(f) \end{matrix} \right]$$

(3·7)

where λ, m, n are positive integers and $2(p+q+r) > (A+B+C+D)$, $2(p+s+t) > (A+B+E+F)$.

(iv) Next on taking $p = A$, $s = E$, $t = 1$, $f_1 = 0$ and make $z \to 0$ in (3·7), then on replacing $A+C$ by A, $B+D$ by B, $A+q$ by s along with necessary changes in parameters, we get following series for G-function :

$$\frac{(1+\alpha)_n \, \Gamma(\beta+1) \, n! \, (-1)^n}{\Gamma(n+\beta+1)} \sum_{m=0}^{n} \frac{(-1)^m \, (n+\alpha+\beta+1)_m}{(1+\alpha)_m \, m! \, (n-m)!} \times$$

$$\times \, G^{r,\, s+\lambda}_{A+\lambda,\, B+\lambda} \left[y \; \Bigg| \; \begin{matrix} \triangle(\lambda, 1-\rho-m),\, (a) \\ (b),\, \triangle(\lambda, -\rho-\beta-m) \end{matrix} \right]$$

$$= G^{r,\, s+2\lambda}_{A+2\lambda,\, B+2\lambda} \left[y \; \Bigg| \; \begin{matrix} \triangle(\lambda, 1-\rho+\alpha),\, \triangle(\lambda, 1-\rho),\, (a) \\ (b),\, \triangle(\lambda, 1-\rho+\alpha+n),\, \triangle(\lambda, -\rho-\beta-n) \end{matrix} \right]$$

(3·8)

where λ, m, n, are positive integers and $2(r+s) > (A+B)$.

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4. *Expansion formula* :

The required formula is

$$x^\rho \, G^{p,\,q,\,s,\,r,\,t}_{A,\,[C,\,E],\,B,\,[D,\,F]} \left[\begin{array}{c} yx^\lambda \\ zx^\lambda \end{array} \middle| \begin{array}{c} (a) \\ (c) \; ; \; (e) \\ (b) \\ (d) \; ; \; (f) \end{array} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n \, \Gamma(\alpha + \beta + n + 1)}{\Gamma(\alpha + n + 1) \, \lambda^{\beta+1}} \times$$

$$\times \, G^{p+2\lambda,\,q,\,s,\,r,\,t}_{A+2\lambda,\,[C,E],\,B+2\lambda,\,[D,F]} \left[\begin{array}{c} y \\ z \end{array} \middle| \begin{array}{c} \Delta(\lambda, -\rho), \; \Delta(\lambda, -\rho-\alpha), \, (a) \\ (c) \; ; \; (e) \\ (b), \; \Delta(\lambda, 1+\rho-n), \; \Delta(\lambda, \beta+\rho+\alpha+n+2) \\ (d) \; : \; (f) \end{array} \right] \times$$

$$\times \, P_n^{(\alpha,\beta)} (1 - 2x) \, \Gamma(\alpha + \beta + 2n + 1) \tag{4.1}$$

The formula (4.1) holds if $R(\rho) > 0$, $R(\alpha) > -1$, $R(\beta) > -1$, $\lambda > 0$, $2p + q + r > (A + B + C + D)$, $2(p + s + t) > (A + B + E + F)$,

$$|\arg y| < \left(p + q + r - \frac{A}{2} - \frac{B}{2} - \frac{C}{2} - \frac{D}{2} \right) \pi, \quad \text{and}$$

$$|\arg z| < \left(p + s + t - \frac{A}{2} - \frac{B}{2} - \frac{E}{2} - \frac{F}{2} \right) \pi.$$

Proof :

$$\text{Let } f(x) = x^\rho \, G^{p,\,q,\,s,\,r,\,t}_{A,\,[C,\,E],\,B,\,[D,\,F]} \left[\begin{array}{c} yx^\lambda \\ zx^\lambda \end{array} \middle| \begin{array}{c} (a) \\ (c) \; ; \; (e) \\ (b) \\ (d) \; ; \; (f) \end{array} \right]$$

$$= \sum_{n=0}^{\infty} R_n \, P_n^{(\alpha,\beta)} (1 - 2x) \tag{4.2}$$

Equation (4.2) is valid, since $f(x)$ is continuous and of bounded variation in the open interval (0, 1) when $y \geqslant 0$, $z \geqslant 0$.

Multiplying both the sides of (4.2) by $x^\alpha (1 - x)^\beta P_u^{(\alpha,\beta)} (1 - 2x)$ and integrating with respect to x from 0 to 1, we have

$$\int_0^t x^{\rho+\alpha} (1 - x)^\beta \, P_u^{(\alpha,\beta)} (1 - 2x) \, G \left[\begin{array}{c} yx^\lambda \\ zx^\lambda \end{array} \right] dx$$

$$= \sum_{n=0}^{\infty} R_n \int_0^1 x^\alpha (1 - x)^\beta \, P_n^{(\alpha,\beta)} (1 - 2x) \, P_u^{(\alpha\beta)} (1 - 2x) \, dx \tag{4.3}$$

using the orthogonal property for Jacobi polynomials [(6), p. 285] and (3.6) to

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evaluate the left hand integral, we have

$$R_n = \frac{(-1)^n \, \Gamma\,(\alpha + \beta + n + 1)\,(\alpha + \beta + 2n + 1)}{\Gamma(\alpha + n + 1)\,\lambda\beta^{+1}} \times$$

$$G^{p+2\lambda,\ q,\ s,\ r,\ t}_{A+2\lambda,\ [C,E],\ B+2\lambda,\ [D,F]}\left[\begin{array}{c} y \\ z \end{array} \middle| \begin{array}{c} \Delta(\lambda, -\rho),\ \Delta(\lambda, -\rho - \alpha),\ (a) \\ (c)\ \ ;\ \ (e) \\ (b),\ \Delta(\lambda,\ 1+\rho - n),\ \Delta(\lambda,\ \rho+\alpha+\beta+n+2) \\ (d)\ \ ;\ \ (f) \end{array}\right] \quad (4.4)$$

After substituting the value of R_n in (4.2) we get the required expansion formula (4.1).

Particular case :   Using (1.5) and on making the appropriate changes in parameters in the equation (4.1), we get the following new expansion formula for G-function :

$$x^\rho\, G^{r,\ s}_{A,\ B}\left[yx^\lambda \middle| \begin{array}{c} (a) \\ (b) \end{array}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n \, \Gamma(\alpha+\beta+n+1)\,(\alpha+\beta+2n+1)}{\Gamma(\alpha + n + 1)\,\lambda\beta^{+1}} \times$$

$$G^{r,\ s+2\lambda}_{A+2\lambda,\ B+2\lambda}\left[y \middle| \begin{array}{c} \Delta(\lambda, -\rho),\ \Delta(\lambda, -\rho - \alpha),\ (a) \\ (b),\ \Delta(\lambda, -\rho + n),\ \Delta(\lambda, -\rho-\alpha-\beta-n-1) \end{array}\right] P_n^{(\alpha,\beta)}\,(1-2x) \quad (4.5)$$

with the conditions easily obtainable from (4.1).

Acknowledgement

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Chemical Study of the Seed of *Pterospermum acerifolium*

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Abstract

In our previous communication[1] we have reported the presence of β–amyrin in the unsaponifiable matter of the oil isolated from the seed of *Pterospermum acerifolium*[2,3]. The present paper deals with the study of a glycoside isolated from the alcoholic extract of the seeds of *Pterospermum acerifolium*. In addition to this glycoside, the presence of some amino acids (tyrosine, cystine, glycine and alanine) and sugars (lactose, xylose, rhamnose and glucose) has also been confirmed.

The ethanolic extract of the seed of *Pterospermum acerifolium* on concentration under reduced pressure gave a semi-solid sticky mass. Solvent ether and benzene extraction of the sticky mass removed fatty material and chlorophyll, and the mass was then extracted with hot water and filtered. The filtrate (A) and the residue (B) were studied separately.

Study of the Filtrate (A)

10 ml. of the water extract were centrifuged and then chromatographed on Whatman No. 1 filter paper using *n*–butanol : acetic acid : water (4 : 1 : 5) as solvent and ninhydrin as spray reagent. The Rf value of the spots obtained on comparison with those given in the literature[4] were found to correspond with glycine, tyrosine, cystine and alanine. The water extract was then treated with lead acetate when a precipitate (lead-lake) appeared, which was separated by filtration.

Study of the Lead-Lake

The lead-lake was suspended in alcohol (100 ml.) and decomposed with H_2S, and the precipitate filtered off. The filtrate when concentrated under reduced pressure (to 20 ml.) gave positive test for the presence of a glycoside, and hence was hydrolysed with 2 N sulphuric acid for 6 hours on a water bath. The contents when poured into a 250 ml. beaker containing ice-cold water gave a white precipitate (aglycon) which was separated by filtration. Paper chromatography of the hydrolysate revealed the presence of glucose as sugar moiety of the glycoside.

Study of Aglycon

The aglycon was found to be insoluble in petrolium ether but soluble in benzene, chloroform, acetone and methanol. On TLC it gave a single spot indicating it to be a pure compound. It was crystallised from benzene : solvent ether (1 : 1) and was obtained in fine white needles (m. p. 197–200°C).

Elemental analysis and molecular weight determination of the compound gave the following results :

<div style="display:flex">
<div>

Found

C = 84·38%

H = 11·85

Percentage of C–CH$_3$ group = 26.

Molecular weight = 417
(Semi–micro Rast)

$[\alpha]_D^{250}$ = + 87·5 (CHCl$_3$)

</div>
<div>

Calculated for C$_{30}$H$_{50}$O

C = 84·51%

H = 11·74%

Calculated for 8 (C–CH$_3$) groups = 28.

Molecular weight = 426

$[\alpha]_D^{250}$ = + 85·5 (CHCl$_3$) for β-amyrin

</div>
</div>

It gave a red colour in Liebermann–Burchard reaction, purple colour with thionyl chloride (Noller's reagent) and yellow colour with tetra-nitromethane showing it to be a triterpene[4]. It formed an acetyl derivative (needles from ethanol–ether) C$_{32}$H$_{52}$O$_2$, m.p. 236°C (acetyl group = 12·9%) and benzoyl derivative C$_{37}$H$_{54}$O$_2$, m.p. 229–30°, indicating that it contained one –OH group.

In literature[5,6,7] the melting points of β–amyrin, its acetyle derivative and benzoyl derivative have been reported to be 199–200°, 236°, and 229–30° respectively. These values correspond to the melting points of the substance isolated and its acetyl and benzoyl derivatives. The melting point of the compound was not depressed when mixed with the authentic sample of β-amyrin.

The infra-red spectrum of the compound shows peaks at 3220 cm^{-1} (–OH), 2900 cm^{-1} (–CH$_3$, –CH$_2$), 1470 cm^{-1} (–C–CH$_3$), 1380 cm^{-1}, 820 cm^{-1}, 1000 cm^{-1} (cyclohexane) and further confirms its identity with β-amyrin.

Study of the Lead–Lake Filtrate

The lead-lake filtrate was treated with ammonia, when a sticky precipitate appeared and was filtered. The chemical study of this precipitate was not possible due to its small amount.

The lead present in the ammoniacal filtrate was removed by passing H$_2$S and filtering off the precipitate. The solution was concentrated under reduced pressure and the semi-solid residue thus obtained was found to reduce Fehling's solution as well as ammoniacal silver nitrate. Paper chromatography using n-butanol : acetic acid : water (4 : 1 : 5) as solvent and aniline hydrogen phthalate as spray reagent revealed the presence of glucose, rhamnose and xylose.

Study of the Residue (B)

The water insoluble ethanolic residue was treated with acetone. The acetone extract gave violet colour with ferric chloride indicating the presence of phenolic –OH group in it (ν_{max}^{Nujol} 3400 cm^{-1}). The solvent was evaporated completely under reduced pressure and the residue thus obtained was chromatographed on a TLC plate. Though it showed only single spot on thin-layer chromatography, it could not be crystallised due to its amorphous nature. It was found to be soluble in acetone, chloroform, carbon tetra-chloride, methanol and ethanol, but insoluble in ether and benzene. It melts at 112–116° and does not contain N, S or P. It neither responded the test for the presence of tri-terpenes, nor for the sterols.

Found	*Calculated for* $C_{12}H_{20}O_2$
C = 74·06%	C = 73·46%
H = 9.69%	H = 10·20%
O = 16·25%	O = 16·34%
Molecular weight = 188	Molecular weight = 196
(Semi–micro Rast)	

It did not reduce the Fehling's solution but reduced ammoniacal silver nitrate and gave colour with Schiff's reagent thus indicating the presence of aromatic aldehyde group in the compound (ν_{max}^{Nujol} 1700 cm⁻¹, 1306 cm⁻¹, 1276 cm⁻¹).

The peaks in the infra-red spectrum at frequencies (ν_{max}^{Nujol} 2900 cm⁻¹, 2850 cm⁻¹, 1470 cm⁻¹, 1380 cm⁻¹ and 1168 cm⁻¹) indicate the presence of –OCH₃, C–CH₃, –CH₂ and –C(CH₃)₂ groups respectively.

Further study of the compound is in progress.

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Results involving Hermite polynomials of two variables

By

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Abstract

For Hermite polynomials of two variables, as adopted by Parashar, certain results are given in this paper.

1. Introduction

Following Ditkin and Prudnikov [2], Parashar [8] has recently introduced and studied some of the properties of the Hermite polynomials $H_n(x,y)$ in two variables x, y and of integral order n, defined by

$$H_{2n}(x,y) = \frac{(2n)!}{n!} \; \pi \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r}(x) \, y^{2n-2r}}{2^{-r} \, \Gamma(n-r+\frac{1}{2}) \, \Gamma(r+\frac{1}{2})} \;, \tag{1.1}$$

$$H_{2n+1}(x,y) = \frac{(2n+1)!}{n!} \; \pi \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r+1}(x) \, y^{2n-2r+1}}{2^{2r+1} \, \Gamma(n-r+\frac{3}{2}) \, \Gamma(r+\frac{3}{2})} \;, \tag{1.2}$$

with generating functions

$$e^{t^2} \cos 2xt \, \cos 2yt = \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n}(x,y) \, t^{2n}}{(2n)!} \;, \tag{1.3}$$

and

$$e^{t^2} \sin 2xt \, \sin 2yt = \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}(x,y) \, t^{2n+2}}{(2n+1)!} \;, \tag{1.4}$$

valid for all finite values of x, y and t.

For Hermite polynomials, $H_n(x)$, of one variable,

we have $\qquad e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} \;.$ (1.5)

Various properties of $H_n(x)$ are well known [3, 4, 9].

2. Results

(1.3) and (1.4) with (1.5) give

$$H_{2n}(x,y) = \tfrac{1}{2} \{ H_{2n}(x+y) + H_{2n}(x-y) \}, \tag{2.1}$$

$$H_{2n+1}(x,y) = \tfrac{1}{2} \{ H_{2n+2}(x+y) - H_{2n+2}(x-y) \} \tag{2.2}$$

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It is interesting to note that in view of the generating function $e^{t^2} \sin 2xt$ $\cos 2yt$, second type of Hermite polynomials $H_{2n+1}^{(2)}(x,y)$ may be defined as the coefficient of $\dfrac{(-1)^n t^{2n+1}}{(2n+1)!}$ in the expansion. Thus

$$e^{t^2} \sin 2xt \cos 2yt = \sum_{n=0}^{\infty} \frac{(-1)^n H_{2n+1}^{(2)}(x,y)\, t^{2n+1}}{(2n+1)!} \qquad (2.3)$$

In turn it is seen that

$$H_{2n+1}^{(2)}(x,y) = \tfrac{1}{2}\{\, H_{2n+1}(x+y) + H_{2n+1}(x-y)\,\}, \qquad (2.4)$$

$$H_{2n+1}^{(2)}(y,x) = \tfrac{1}{2}\{\, H_{2n+1}(x+y) - H_{2n+1}(x-y)\,\}. \qquad (2.5)$$

This shows that $H_n(x,y)$ adopted by Parashar and $H_{2n+1}^{(2)}(x,y)$ defined above can be easily studied by means of the Hermite polynomials of one variable. Recurrence relations, integral representations and addition formulae are its easy consequences. For further results, we have

$$H_{2n}(x,y) = \frac{(2n)!}{2^n} \sum_{r=0}^{n} \frac{H_{2r}(\sqrt{2}\,x)\, H_{2(n-r)}(\sqrt{2}\,y)}{(2r)!\, 2(n-r)!} \qquad (2.6)$$

$$H_{2n+1}(x,y) = \frac{(2n+1)!}{2^{n+1}} \sum_{r=0}^{n} \frac{H_{2r+1}(\sqrt{2}\,x) H_{2(n-r)+1}(\sqrt{2}\,y)}{(2r+1)!\,(2n-2r+1)!}, \qquad (2.7)$$

$$H_{2n+1}^{(2)}(x,y) = \frac{(2n+1)!}{2^{n+1}} \sum_{r=0}^{n} \frac{H_{2r+1}(\sqrt{2}\,x)\, H_{2(n-r)}(\sqrt{2}\,y)}{(2r+1)!\, 2(n-r)!}, \qquad (2.8)$$

and

$$H_{2n+1}^{(2)}(x,y) = \frac{(2n+1)!}{n!}\pi \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r+1}(x)\, y^{2n-2r}}{2^{2r+1}\, \Gamma(n-r+\tfrac{1}{2})\, \Gamma(r+\tfrac{3}{2})} \qquad (2.9)$$

(1.1), (1.2) and (2.9) suggest to observe whether an allied function may be adopted by means of the expression

$$\frac{(2n)!}{n!}\pi \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r}(x)\, y^{2n-2r+1}}{2^{2r}\, \Gamma(n-r+\tfrac{3}{2})\, \Gamma(r+\tfrac{1}{2})}.$$

It is found that the above expression is equivalent to $H_{2n+1}^{(2)}(y,x) / (2n+1)$. Therefore, we have

$$H_{2n+1}^{(2)}(y,x) = \frac{(2n+1)!}{n!}\pi \sum_{r=0}^{n} \binom{n}{r} \frac{H_{2r}(x)\, y^{2n-2r+1}}{2^{2r}\, \Gamma(n-r+\tfrac{3}{2})\, \Gamma(r+\tfrac{1}{2})}. \qquad (2.10)$$

As an example, with the help of [1]

$$\int_{-\infty}^{\infty} e^{-x^2}\, H_m(x)\, H_n(x)\, dx = 2^n\, n!\, \sqrt{\pi}, \quad \text{when } m = n$$
$$= 0, \qquad \text{when } m \neq n;$$

(2.6) – (2.8) give

$$\int_{-\infty}^{\infty} e^{-2x^2}\, H_{2n}(x,x)\, dx = 2^{2n}\, \Gamma(n+\tfrac{1}{2}), \quad \text{when } n = 0, 2, 4, \ldots$$
$$= 0, \qquad \text{when } n = 1, 3, 5, \ldots; \qquad (2.11)$$

$$\int_{-\infty}^{\infty} e^{-2x^2} H_{2n+1}(x,x)\, dx = 2^{2n+1}\, \Gamma\left(n + \tfrac{3}{2}\right), \text{ when } n = 0, 2, 4, \ldots$$

$$= 0, \qquad\qquad\qquad \text{when } n = 1, 3, 5, \ldots; \quad (2\cdot12)$$

$$\int_{-\infty}^{\infty} e^{-2x^2} H_{2n+1}{}^{(2)}(x,x)\, dx = 0, \qquad\qquad \text{when } n = 1, 2, 3, \ldots. \quad (2\cdot13)$$

Similarly, it is found that

$$\int_{-\infty}^{\infty} e^{-4x^2} H_{2m}(x,x)\, H_{2n}(x,x)\, dx = 2^{2n-3}(2n)!\,\sqrt{\pi} + 2^{-\frac{7}{2}}\left(\frac{2n!}{n!}\right)^2\sqrt{\pi}, \text{ when } m=n$$

$$= 2^{-\frac{7}{2}}(-1)^{m+n}\,\frac{(2m)!\,(2n)!}{m!\,n!}\,\sqrt{\pi}, \text{ when } m \neq n; \quad (2\cdot14)$$

$$\int_{-\infty}^{\infty} e^{-4x^2} H_{2m+1}(x,x)\, H_{2n+1}(x,x)\, dx = 2^{2n-1}(2n+2)!\,\sqrt{\pi} + 2^{-\frac{7}{2}}\left(\frac{(2n+2)!}{(n+1)!}\right)^2\sqrt{\pi}, \text{ when } m=n$$

$$= 2^{-\frac{7}{2}}(-1)^{m+n}\,\frac{(2m+2)!\,(2n+2)!}{(m+1)!\,(n+1)!}\,\sqrt{\pi}, \text{ when } m \neq n; \quad (2\cdot15)$$

$$\int_{-\infty}^{\infty} e^{-4x^2} H_{2m+1}{}^{(2)}(x,x)\, H_{2n+1}{}^{(2)}(x,x)\, dx = 2^{2n-2}(2n+1)!\,\sqrt{\pi}, \text{ when } m=n,$$

$$= 0, \text{ when } m \neq n. \quad (2\cdot16)$$

In case we use [4]

$$\int_{-\infty}^{\infty} e^{-x^2} H_{2n}\left(\frac{x}{\sqrt{2}}\right)\, dx = (-1)^n\, 2^n\, \Gamma\left(n + \tfrac{1}{2}\right),$$

we find that

$$\int_{-\infty}^{\infty} e^{-8x2} H_{2n}(x,x)\, dx = (-1)^n\, 2^{n-\frac{5}{2}}\,(2^n + 1)\,\Gamma\left(n + \tfrac{1}{2}\right), \quad (2\cdot17)$$

$$\int_{-\infty}^{\infty} e^{-8x^2} H_{2n+1}(x,x)\, dx = (-1)^n\, 2^{n-\frac{3}{2}}\,(2^{n+1} - 1)\,\Gamma\left(n + \tfrac{3}{2}\right), \quad (2\cdot18)$$

$$\int_{-\infty}^{\infty} e^{-8x^2} H_{2n+1}{}^{(2)}(x,x)\, dx = 0, \text{ when } n = 0, 1, 2, \ldots. \quad (2\cdot19)$$

Relations cited above shall give more interesting results.

By means of generating functions one can easily associate the Hermite polynomials of two variables, given above, with the generalised Bessel coefficients [5] and determinants occurring in generalised exponential expansions [6, 7].

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On some n-th order finite difference inequalities

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1. A useful tool in obtaining bounds for solutions of differential equations is the Gronwall inequality, also known as Bellman's Lemma and its various generalizations. The content of these theorems is to compare solutions of certain inequalities with solutions of the corresponding equations. The object of this note is to establish, similar results for n-th order finite difference equations, which are useful in comparing the solutions of certain n-th order finite difference inequalities with those of the solutions of the corresponding n-th order finite difference equations, without reducing it to a system of n equations of the first order.

2. Let G be an open set in R^n, an n-dimensional vector-space. We shall denote the n-tuple $(x_1, x_2, \ldots, x_n)$ as an element of R^n. Let $f : I \times G \to R^n$ be a continuous function, where I denotes the integers $o \leqslant k < \infty$. Further I_k^N denotes the set of positive integers from k to N for $o \leqslant k \leqslant N$. Consider the finite difference equation of n-th order with initial conditions

$$\triangle^{(n)} x(k) = f(k, x(k), \triangle x(k), \ldots, \triangle^{(n-1)} x(k)),$$

$$\triangle^{(i)} x(o) = a_i, \text{ for each } i \, \varepsilon \, I_0^{(n-1)} \qquad (2 \cdot 0)$$

where $\triangle^{(i)} x(k) = \triangle^{(i-1)} x(k+1) - \triangle^{(i-1)} x(k)$. It is easy to see that for every point P in $I \times G$ there is a unique solution $x(k)$ of $(2 \cdot 0)$ existing on I. One can view that the function f defined on $I \times G$ as the restriction of a continuous function defined on $R' \times G$, where R' denotes the real numbers. This viewpoint will be assumed in sequal.

Our discussion depends on the following fundamental theorem on n-th order finite difference inequalities.

Theorem 1 : Let the scalar function $f : I \times G \to R'$ be continuous and monotonically increasing in $x_1, x_2, \ldots, x_{n-1}$. Let $x(k)$ and $y(k)$ be real valued continuous functions defined on I such that $\triangle^{(n-1)} x(k)$, $\triangle^{(n-1)} y(k)$ exists for all $k \, \varepsilon \, I$ and $(k, x(k), \triangle x(k), \ldots, \triangle^{(n-1)} x(k))$, $(k, y(k), \triangle y(k), \ldots, \triangle^{(n-1)} y(k))$ $\varepsilon \, I \times G,$ and

$$\triangle^{(i)} x(o) \leqslant \triangle^{(i)} y(o), \text{ for each } i \, \varepsilon \, I_0^{(n-1)}.$$

Assume further that the inequalities

$$\triangle^{(n)} x(k) \leqslant f(k, x(k), \triangle x(k), \ldots, \triangle^{(n-1)} x(k)), \qquad (2 \cdot 1)$$

[235]

$$\Delta^{(n)} y(k) \geqslant f(k, y(k), \Delta y(k), \ldots, \Delta^{(n)} y(k))$$ (2·2)

hold for $k \, \varepsilon \, I$. Then

$$x(k) \leqslant y(k)$$ (2·3)

for all $k \, \varepsilon \, I$.

Proof. If the assertion (2·3) is false, then the set

$$S_0 = \{\, k : y(k) < x(k), \, k > 0 \,\}$$

is not empty. Define $k_0 = \min S_0$, for which the inequality (2·3) is not true. Then we will have

$$y(k_0) < x(k_0)$$ (2·4)

and

$$x(k_0 - 1) \leqslant y(k_0 - 1).$$ (2·5)

Using (2·4) and (2·5) we obtain

$$\Delta y(k_0 - 1) < \Delta x(k_0 - 1).$$

This means that the set

$$S_1 = \{\, k : \Delta y(k) < \Delta x(k), \, k > 0 \,\}$$

is also not empty. Assume that the set S_j defined by

$$S_j = \{\, k : \Delta^{(j)} y(k) < \Delta^{(j)} x(k), \, k > 0 \,\}, \, 1 \leqslant j < n - 1,$$

is not empty. As before we define $k_j = \min S_j$. Then we will have

$$\Delta^{(j)} y(k_j) < \Delta^{(j)} x(k_j)$$ (2·6)

and

$$\Delta^{(j)} x(k_j - 1) \leqslant \Delta^{(j)} y(k_j - 1).$$ (2·7)

Using (2·6) and (2·7) we have

$$\Delta^{(j+1)} y(k_j - 1) < \Delta^{(j+1)} x(k_j - 1),$$ (2·8)

which means that the set S_{j+1} defined by

$$S_{j+1} = \{\, k : \Delta^{(j+1)} y(k) < \Delta^{(j+1)} x(k), \, k > 0 \,\}$$

is also not empty. Now by the method of mathematical induction and the assumption that S_0 is non-empty, we conclude that $S_1, S_2, \ldots, S_{n-1}$ are also non-empty. Note that $k_{j+1} \leqslant k_j, j \, \varepsilon \, I_0^{(n-1)}$. Now following (2·6), (2·7) and (2·8) we can similarly write

$$\Delta^{(n-1)} y(k_{n-1}) < \Delta^{(n-1)} x(k_{n-1})$$ (2·9)

and

$$\Delta^{(n-1)} x(k_{n-1} - 1) \leqslant \Delta^{(n-1)} y(k_{n-1} - 1)$$ (2·10)

$$\Delta^{(n)} y(k_{n-1} - 1) < \Delta^{(n)} x(k_{n-1} - 1).$$ (2·11)

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The relations $(2\cdot1)$, $(2\cdot2)$ and $(2\cdot11)$ yields the inequality

$$f(k_{n-1} - 1, x(k_{n-1} - 1), \triangle x(k_{n-1} - 1), \ldots, \triangle^{(n-1)} x(k_{n-1} - 1))$$

$$> f(k_{n-1} - 1, y(k_{n-1} - 1), \triangle y(k_{n-1} - 1), \ldots, \triangle^{(n-1)} y(k_{n-1} - 1)).$$

Note that $k_0 \geqslant k_i \geqslant k_{n-1} > 0$, for $i \ \varepsilon \ I_1^{(n-2)}$.

This is a contradiction in view of the monotonic property of f and $(2\cdot10)$. Hence the set S_{n-1} is empty. Following the same argument we can show that the sets $S_{n-2}, \ldots, S_1$ and S_0 are also empty; which establishes $(2\cdot3)$.

Remark. If in theorem 1, the inequalities $(2\cdot1)$ and $(2\cdot2)$ be replaced by the inequality

$$\triangle^{(n)} x(k) - f(k, x(k), \triangle x(k), \ldots, \triangle^{(n-1)} x(k))$$

$$\leqslant \triangle^{(n)} y(k) - f(k, y(k), \triangle y(k), \ldots, \triangle^{(n-1)} y(k))$$

for each $k \ \varepsilon \ I$, then, the conclusion $(2\cdot3)$ of theorem 1 remains valid.

Definition. Let $x(k)$ be a continuous function defined on I such that $\triangle^{(n)} x(k)$ exists fot $k \ \varepsilon \ I$. If $x(k)$ satisfies the difference inequality

$$\triangle^{(n)} x(k) < f(k, x(k), \triangle x(k), \ldots, \triangle^{(n-1)} x(k)), \ k \ \varepsilon \ I.$$

it is said to be an under-function with respect to $(2\cdot0)$. On the other hand if

$$\triangle^{(n)} x(k) > f(k, x(k), \triangle x(k), \ldots, \triangle^{(n-1)} x(k)), \ k \ \varepsilon \ I,$$

$x(k)$ is said to be an over–function.

The following theorem shows that any solution of $(2\cdot0)$ can be bracketed between its under– and over– function.

Theorem 2. Let $y(k)$, $z(k)$ be under- and over-functions with respect to $(2\cdot0)$ respectively on I. If $x(k)$ is any solution of $(2\cdot0)$ existing on I, such that

$$\triangle^{(i)} y(o) < a_i < \triangle^{(i)} z(o), \text{ for each } i \ \varepsilon \ I_0^{(n-1)}.$$

Then

$$y(k) < x(k) < z(k), \text{ for } k \ \varepsilon \ I.$$

The proof follows by repeated applications of Theorem 1. We omit the details.

3. We formulate the following comparison theorem, which is an extension of Theorem 1 in [10], to n-th order finite difference equations, which in itself is the generalization of Bellman's Lemma [2].

Theorem 3. Let the scalar function $W(k, x_1, x_2, \ldots, x_{n-1}) \geqslant 0$ be defined and continuous on $I \times R'$ and monotonically increasing in $x_1, x_2, \ldots, x_{n-1}$ for each $k \ \varepsilon \ I$ and satisfy the inequality,

$$\triangle^{(n)} m(k) \leqslant W(k, m(k), \triangle m(k), \ldots, \triangle^{(n-1)} m(k)), \ k \ \varepsilon \ I.$$

[237]

Let $r(k)$ be the solution of

$$\Delta^{(n)} r(k) = W(k, r(k), \Delta r(k), \ldots, \Delta^{(n-1)} r(k));$$

$$\Delta^{(i)} r(o) = c_i \text{ for each } i \in I_0^{(n-1)},$$ (3·1)

existing on I, such that

$$\Delta^{(i)} m(o) \leqslant c_i, \text{ for each } i \in I_0^{(n-1)}.$$

Then

$$m(k) \leqslant r(k), \text{ for } k \in I.$$

The proof of this theorem follows by a direct application of Theorem 1.

We give another comparison theorem which, in certain situations, is more useful than the Theorem 3.

Theorem 4. Let the function W be defined as in Theorem 3, and $r(k)$ be the solution of (3·1) existing on I. Consider two n-th order finite difference equations

$$\Delta^{(n)} x(k) = f(k, x(k), \Delta x(k), \ldots, \Delta^{(n-1)} x(k));$$

$$\Delta^{(i)} x(o) = a_i, \; i \in I_0^{(n-1)}.$$ (3·2)

$$\Delta^{(n)} y(k) = g(k, y(k), \Delta y(k), \ldots, \Delta^{(n-1)} y(k));$$

$$\Delta^{(i)} y(o) = b_i, \; i \in I_0^{(n-1)},$$ (3·3)

where f and g are continuous scalar functions defined on $I \times G \to R'$ and satisfy the condition

$$\| f(k, x(k), \Delta x(k), \ldots, \Delta^{(n-1)} x(k)) - g(k, y(k), \Delta y(k), \ldots, \Delta^{(n-1)} y(k)) \|$$
$$\leqslant W(k, \| x(k) - y(k) \|, \| \Delta x(k) - \Delta y(k) \|, \ldots, \| \Delta^{(n-1)} x(k) - \Delta^{(n-1)} y(k) \|) \quad (3·4)$$
for $k \in I$. Let $x(k)$ and $y(k)$ be any two solutions of (3·2) and (3·3) respectively existing on I such that

$$\| a_i - b_i \| \leqslant c_i; \text{ for } i \in I_0^{(n-1)}.$$

Then

$$\| x(k) - y(k) \| \leqslant r(k), \text{ for all } k \in I.$$

Proof. Let $m(k) = \| x(k) - y(k) \|$, then we have

$$\Delta^{(n)} m(k) \leqslant \| f(k, x(k), \Delta x(k), \ldots, \Delta^{(n-1)} x(k)) - g(k, y(k), \Delta y(k), \ldots, \Delta^{(n-1)} y(k)) \|$$
$$\leqslant W(k, \| x(k) - y(k) \|, \| \Delta x(k) - \Delta y(k) \|, \ldots, \| \Delta^{(n-1)} x(k) - \Delta^{(n-1)} y(k) \|)$$

i.e. $\Delta^{(n)} m(k) \leqslant W(k, m(k), \Delta m(k), \ldots, \Delta^{(n-1)} m(k))$, by (3·4).

Now, an application of Theorem 3, gives the desired result.

[238]

4. In this section we state and prove two important comparison theorems, which are the further generalizations of theorem 3 and theorem 4 of the previous section.

Theorem 5. Let the scalar function $W_1(k, x_1, x_2, \ldots, x_{n-1}) \geqslant 0$,

$W_2(k, x_1, x_2, \ldots, x_{n-1}) \geqslant 0$ be defined and continuous on $I \times G$, $G \subset R'$ and monotonically increasing in $x_1, x_2, \ldots, x_{n-1}$, and satisfy the inequality

$$W_2(k, m(k), \triangle m(k), \ldots, \triangle^{(n-1)} m(k))$$

$$\leqslant \triangle^{(n)} m(k) \leqslant W_1(k, m(k), \triangle m(k), \ldots, \triangle^{(n-1)} m(k)) \tag{4.1}$$

for all $k \geqslant 0$. Let $u(k)$ and $v(k)$ be the solutions of

$$\triangle^{(n)} u(k) = W_1(k, u(k), \triangle u(k), \ldots, \triangle^{(n-1)} u(k)),$$

$$\triangle^{(i)} u(o) = d_i, \ i \ \varepsilon \ I_0^{(n-1)}, \tag{4.2}$$

$$\triangle^{(n)} v(k) = W_2(k, v(k), \triangle v(k), \ldots, \triangle^{(n-1)} v(k)),$$

$$\triangle^{(i)} v(o) = e_i, \ i \ \varepsilon \ I_0^{(n-1)}, \tag{4.3}$$

respectively, existing on I, such that

$$e_i \leqslant \triangle^{(i)} m(o) \leqslant d_i, \text{ for } i \ \varepsilon \ I_0^{(n-1)}.$$

Then

$$v(k) \leqslant m(k) \leqslant u(k) \tag{4.4}$$

for all $k \geqslant 0$.

Proof. We shall prove the right half of the inequality (4.4), similar reasoning can be used for the left half. From the inequality (4.1) we have

$$\triangle^{(n)} m(k) \leqslant W_1(k, m(k), \triangle m(k), \ldots, \triangle^{(n-1)} m(k)). \tag{4.5}$$

Since $u(k)$ is the solution of (4.2) we have

$$\triangle^{(n)} u(k) = W_1(k, u(k), \triangle u(k), \ldots, \triangle^{(n-1)} u(k)),$$

$$\triangle^{(i)} u(o) = d_i, i \ \varepsilon \ I_0^{(n-1)}. \tag{4.6}$$

Now, applying Theorem 3 to (4.5) and (4.6) we obtain

$$m(k) \leqslant u(k), \text{ for all } k \geqslant 0.$$

Theorem 6. Let the functions W_1 and W_2 be defined as in theorem 5. Suppose further that the functions f and g of (3.2) and (3.3), satisfy the condition

$$W_2(k, \| x(k) - y(k) \|, \| \triangle x(k) - \triangle y(k) \|, \ldots, \| \triangle^{(n-1)} x(k) - \triangle^{(n-1)} y(k) \|)$$
$$\leqslant \| f(k, x(k), \triangle x(k), \ldots, \triangle^{(n-1)} x(k)) - g(k, y(k), \triangle y(k), \ldots, \triangle^{(n-1)} y(k)) \|$$
$$\leqslant W_1(k, \| x(k) - y(k) \|, \| \triangle x(k) - \triangle y(k) \|, \ldots, \| \triangle^{(n-1)} x(k) - \triangle^{(n-1)} y(k) \|)$$

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for all $k \geqslant 0$. Let $u(k)$ and $v(k)$ be any two solutions of (4·2) and (4·3) respectively. If $x(k)$ and $y(k)$ be the solutions of the equations (3·2) and (3·3) respectively, such that

$$e_i \leqslant \| a_i - b_i \| \leqslant d_i, \quad \text{for } i \, \varepsilon \, I_0^{(n-1)}.$$

Then

$$v(k) \leqslant \| x(k) - y(k) \| \leqslant u(k),$$

for all $k \geqslant 0$.

The proof of this theorem follows by repeated applications of theorem 4.

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